REED RESONANCE EFFECTS
ON WOODWIND NONLINEAR
FEEDBACK OSCILLATIONS

by

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Abstract

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The linear explanation of woodwind oscillations says that an oscillation can occur at any frequency for which the product of the air column input impedance $Z$ and the reed transconductance $A$ is greater than unity ($ZA > 1$), and that any number of these oscillations can occur simultaneously and independently. In actual practice the nonlinear flow control property of the reed transfers energy among the harmonics of the spectrum. For low register notes the oscillation is favored at a frequency for which the input impedance is high at several harmonics of the note thus maximizing the net production of acoustic energy at the harmonics as well as at the playing frequency. However above the middle of the second playing register, woodwinds have only a single participating impedance peak because the instrument's tone hole lattice cutoff frequency is below even the second harmonic of the playing frequency for these notes; nevertheless they can be played even without the use of the register hole despite the possibilities of low register intermode cooperation which seem to favor low register oscillation. The upper notes are possible because of a resonant enhancement of the reed's transconductance near its own resonance frequency $f_r$. This increased $A$ can offset
the smallness of \( Z \) beyond cutoff so that \( ZA^2 \) at a harmonic of the playing frequency providing an additional means of energy production beyond cutoff.

The nonlinear theory developed by Worman is reformulated and the equations rewritten to show more clearly their relationship with linear feedback theory. The assumptions of Worman's theory limit its application to small and medium amplitudes where the reed does not beat. The new formulation permits one to deduce a considerable amount of the behavior of the system from the structure of the new equations without actually solving the equations in detail. The theory is then extended to include those oscillations for which the playing frequency is a sizeable fraction of \( f_r \). In this case it is shown that the oscillation is stabilized if the reed frequency is placed near to but slightly above a low order harmonic of the playing frequency. Experiments under actual playing conditions show that the clarinet player can comfortably adjust his embouchure to change the reed frequency from slightly below 2 kHz to slightly above 3 kHz. This is precisely the range needed to stabilize those notes played on an air column which has only a single impedance peak by providing an additional source of energy at the second or third harmonic of the tone. Experiments on a blowing machine confirm that the most stable oscillations occur when this harmonic matching takes place. Under these conditions the component near the reed frequency is significantly stronger than it would otherwise be, and the heterodyne coupling with the fundamental component also increases the amplitude
of its neighbors. The implications of this theory on the design and playing behavior of woodwinds are discussed with several specific examples cited. The theory is presented in two ways: one in the language of mathematical physics, and the other cast in the technical language of the musician as an aid in his understanding of the phenomena that are important to his profession.
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CHAPTER I

INTRODUCTION

This thesis is a further development and extension of the theory of nonlinear self sustained musical oscillators which was initiated by Benade and Gans\(^1\) and formalized by Worman.\(^2\) In its present form the theory can now describe the steady state oscillation mechanism and general behavior of all notes of the clarinet. Most of the conclusions also apply to the other reed instruments, and with certain minor modifications, to the brass instruments as well. Worman's development formerly had been sufficient only for the low registers of the reed instruments.

A. Reed Instrument Acoustics: A Historical Review

Some of the earliest work which has direct application to reed woodwinds is that done by Weber\(^3\) in about 1830. While his work was mainly concerned with metal reeds on organ pipes, Weber did calculate the natural frequency of an air column terminated by a reed. He found that the natural frequencies of the coupled system are always less than the natural frequency of the reed considered separately from the tube. In fact he showed that there is a natural frequency of the composite system corresponding to each of the natural frequencies of a tube rigidly closed at the reed end; and that the air column modes whose frequencies are far below the reed frequency have
their natural frequencies lowered only very slightly from the values for a rigidly closed tube, while those with frequencies far above the reed frequency all become nearly degenerate just below the reed frequency. These calculations assumed that both the reed and air column behave as undamped linear oscillators. Weber did not deal with the regeneration mechanism required to maintain the oscillations, although he did observe experimentally that certain combinations of reed and air column natural frequencies would not sustain an oscillation. Weber's main contribution was to show that, far below the reed resonance frequency, the natural frequencies of the air column are slightly lowered due to the elastic termination of the air column provided by the reed.

The next major advance in our understanding of reed instruments was made by Helmholtz. In an appendix to his book Sensations of Tone, Helmholtz showed that the regeneration mechanism places restrictions on the relative phase of the oscillations of the reed and air column which can only be satisfied if the playing frequency is either very near the reed frequency or slightly below the natural frequency of an air column mode. This calculation included the effects of reed damping and of radiation damping in the tube, but not the effects of energy loss at the tube walls. Wall losses can, however, be included without changing the general conclusions. Although he did not explicitly state it, Helmholtz's analysis is valid for any air column termination which does not drastically alter the frequencies of the normal modes from those of a rigidly closed
tube. In particular it does apply to the elastic termination provided by the reed far below its resonance frequency. Thus Helmholtz showed that the playing frequencies of a musical instrument are below the natural frequencies of the air column, which Weber had shown to be below the natural frequencies of a rigidly closed tube. It must be emphasized that Helmholtz’s calculations are based on an entirely linear theory; i.e., they assume that the motion of the reed and the air column, and the coupling between them are all linear functions of the generalized coordinates of the system. While the calculations are entirely correct, their applicability to musical systems is limited because a linear theory cannot account for the amplitude and spectrum stability of musical oscillations.

The next contribution to woodwind acoustics came in Lord Rayleigh’s book *The Theory of Sound* which first appeared in 1877 and remains a standard reference today. Of direct importance to musical acoustics (as well as to all of wave physics) is the development of perturbation theory which has allowed the calculation of the effects of small changes in the boundary conditions or in the wave medium itself. Since its development perturbation theory has been used to guide the making of minor changes to the shape of musical air columns to shift their natural frequencies. Using the further developments in the theory of self sustained oscillators made initially by Worman and continued in the present work, perturbation theory can now be used to change the playing behavior of musical instruments in predictable ways. Rayleigh did not add to the
theory of self sustained oscillators, but simply referred to the earlier work by Helmholtz. The next major development in musical acoustics was Dayton Miller's measurement of the spectra of several orchestral instruments. His results were published in his book *The Science of Musical Sounds* in 1916. Also included is an extensive description of his apparatus and experimental method. This book created much interest in the field and many attempts were made to develop theories to explain his spectra. Most of these attempts were made for the clarinet since in some ways it is the simplest acoustical system of the musical instruments. Unfortunately Miller's spectra do not agree with more recent measurements which are known to be valid. The reason for this in part lies in the fact that Miller calibrated his apparatus for flat response to organ tones of equal loudness as judged by the organ builder. He assumed that equal perceived loudness corresponds to equal sound intensity. It has since been verified that for sinusoidal excitations this assumption is not valid. Analysis of Miller's results is further complicated by the fact that the organ pipes used probably had significant higher partials, although Miller did choose to use a set of pipes with the smallest possible harmonic content. It is not known if all of the pipes had the same spectra or if the higher partial amplitudes changed from pipe to pipe. In addition his spectral measurements were made outside of the instrument and on axis which overemphasizes the strengths of the higher components in ways which were not known at the time. It would be interesting to see Miller's experiments
repeated. He was a meticulous experimenter, and there is sufficient information in his book to allow the basic experimental setup to be duplicated and investigated.

In 1929 Henri Bouasse published the results of his work on wind instruments in his two volume book *Instruments à Vent*. Bouasse understood very well the role of the reed in maintaining oscillations. He had achieved considerable physical insight and intuition from his many experiments with all types of wind instruments. While he did not develop any new mathematical theories, he presented many observations which anticipate the nonlinear theory which was later developed by Worman and Benade. For example he stated without explanation the fact that

the maintenance of the standing wave is facilitated by the coincidence of the \(q\)th harmonic of the pressure spectrum due to the [nonsinusoidal air flow through the reed] with the tube mode whose frequency is \(N=qn\) [where \(n\) is the playing frequency]. In this case the [flow] becomes very strong and very stable: One recognizes that the reed motion will be stabilized for the frequency \(n\).

It is interesting to note that Bouasse was apparently not aware of the nonlinear flow control property of the reed, but thought that the harmonic components of the flow were produced because the reed was beating against the tip of the mouthpiece. There is some evidence that he did realize the importance of nonlinear effects because he included as an appendix a standard treatment of the simple "hard spring" oscillator. Unfortunately the work of Bouasse is not widely known, and copies of his works are very hard to obtain. Many of his works, including *Instruments à Vent*, have never been translated from
the French.

The lack of recognition of the work of Bouasse is unfortunate. Bouasse had realized the problems with the spectrum measurements of Dayton Miller and stated that the measurements could not be taken at face value. Without realizing these problems, several other workers proposed theories which attempted to predict Miller's spectra. The agreement in these cases was somewhat forced. In an article by Das\textsuperscript{10} for example, the real part of the air column input impedance was negative. Ghosh\textsuperscript{11} presented a theory which contains several ill-advised assumptions and mathematical errors. No conclusions can be drawn from his treatment. In the second edition of his book \textit{Vibration and Sound}, Philip Morse\textsuperscript{12} presented a similar but mathematically sound treatment of clarinet oscillations. Morse assumed the driving force on the reed to be given by a periodic series of delta functions of alternating sign. At the time this was a reasonable assumption in light of the measurements of McGinnis and Gallagher\textsuperscript{13} who had found that the reed is open for about half of the vibration cycle and closed for the other half. However since that time it has been found that this type of reed motion only occurs at very high playing levels for American style clarinet playing.\textsuperscript{14} For this reason Morse's results are not directly applicable, although the method could possibly be used as the basis for an iterative treatment of beating reed oscillations.

In 1963 John Backus\textsuperscript{15} presented a linear theory of clarinet oscillations which was valid for very small amplitudes near the
threshold of oscillation. He was able to calculate the threshold blowing pressure and the oscillation frequency at threshold. Of more importance to the present work, Backus also made measurements of the flow into the air column as a function of reed tip opening and pressure difference across the reed. As expected he found that the flow is a nonlinear function of both variables, and he determined the form of this function. However he did not make use of this information. Nederveen, in his book Acoustical Aspects of Wind Instruments, presented measurements which show that the flow through double reeds is a different nonlinear function of the pressure difference and reed opening. He extended Backus's theory to the double reed instruments, but again did not include the effects of the nonlinearity. Fairly recently still another small amplitude linear theory has been presented by Wilson and Beavers. They have also presented some experimental results which confirm the earlier work of Bouasse.

The first attempts to include nonlinear effects in the explanation of musical instrument tone production were made in 1958 by Benade in a series of reports for the C. G. Conn company. This work was continued by Benade and Gans and first formally reported in 1964. It was later published in 1968. The continued development of this theory has led to the present work. Also in 1968 in a paper delivered to the Acoustical Society, Pyle presented a nonlinear theory of brass instrument oscillations which involved the numerical solution of integral-differential equations. The initial results
looked promising, but unfortunately the work has not been continued. Fletcher\textsuperscript{22} has developed a similar theory for organ flue pipes which has been able to predict the transient and steady state pressure spectra in good agreement with measured spectra. His theory depends upon the fact that the flow into the flue pipe is controlled by the standing wave velocity at the tube mouth. It cannot be applied to the reed instruments whose flow is controlled by a pressure operated reed. Quite recently Schumacher\textsuperscript{23-25} has developed an integral equation theory for the bowed string instruments and has extended it to include both organ flue pipes and reed instruments. His initial results agree with those of other workers, and the method shows promise of developing new insights into the problem.

In his Ph.D. thesis of 1971 Worman\textsuperscript{26} formalized the Benade–Gans theory for those reed instruments for which the Bernoulli force on the reed tip can be neglected. He was able to solve the coupled nonlinear algebraic equations of the theory for a single simple case to show the validity of the theory. Benade has applied the same general method to the oscillations of the brasses and the bowed strings as well as extending its application in woodwinds.\textsuperscript{27,28} The theory has been successful in explaining many aspects of musical instrument behavior which had not been explained by earlier linear theories. For example the observation of Bouasse that an oscillation is stabilized when a harmonic of the playing frequency coincides with a mode of the tube is readily explained by the nonlinear coupling of harmonically related tube modes which is a consequence of the theory.
This and many other results of Worman's theory are presented clearly and with very little mathematics in Benade's recent book *Fundamentals of Musical Acoustics.*

Throughout this development Worman and Benade have made an assumption which is shared by all previous theories. They have assumed that the natural frequency of the reed is sufficiently high above the playing frequency that the reed resonance does not play an active role in the regeneration process. All of the previous investigators had realized the possibility of oscillation just below the reed frequency, and Bouasse discussed musical oscillations of this type. However for oscillations based on an air column mode, they all assumed that the reed frequency is far above the playing frequency. The present work shows that in fact the reed resonance can play a dynamically significant role in maintaining the oscillations in the upper registers of reed instruments. This occurs when the reed frequency is adjusted to match the frequency of a low order harmonic of the playing frequency. To show this the nonlinear equations developed by Worman are first reformulated in a way which makes it much easier to deduce the effects of changes in the reed and bore parameters of the instrument. This reformulation makes it possible to understand the regeneration mechanism in the upper ranges of reed instruments and thus to predict their behavior. A set of experiments is described which supports the conclusions of the theory.

Along somewhat different lines, Bariaux is developing a method of solution for reed instruments when the reed beats and is
rigidly closed for a part of the cycle. This is very important in understanding the many instruments whose reed beats at low playing levels; among them are the bassoon, the oboe, and the French style clarinet. In addition, the reeds of all instruments beat at loud playing levels. The theory developed in the present work does not hold when the reed beats, and it will be interesting to see the results of this work as it progresses. The initial results of Bariaux for the beating reed show many similarities with the results presented here for the nonbeating reed. In particular he finds that "... the player sets up a cooperation between the various oscillations of the column of air, and then between the regime obtained and the natural frequencies of the reed."31

B. Nonmathematical Explanation of the Theory

The science of musical acoustics is of little value if it cannot accomplish two things. Since the craft of building good instruments and the art of playing them well have existed much longer than the science which attempts to describe them, any valid acoustical theory must successfully explain those practices which musicians have empirically found useful. In addition, if the theory has practical value it will go on to provide the craftsman with ways to improve his instruments and the musician with techniques to improve his playing.

It is only in the last ten years that woodwind acoustics has progressed to the point where the musician finds that he can directly benefit from the work of the scientist. The technique of using
perturbation theory as a guide in making changes in the bore of an instrument to alter the natural frequencies of the air column has been known for a hundred years. However initial attempts to use perturbation theory in the tuning of instruments met with little success because it was not realized that several of the air column modes were involved in the production of each note. As a result, perturbations intended to produce changes in the playing frequency often failed to produce the desired result and also changed the tone color and stability of the note in unpredictable ways. In addition the properties of other notes were often affected unexplainably. The information provided by Worman's nonlinear theory has allowed perturbation theory to be used to adjust the frequency relations between the impedance peaks in a way which gives consistently good results for the low register notes of woodwinds. The present report extends that work to show that the natural frequency of the reed plays a major role in determining the properties of the notes in the upper register.

At very small amplitudes the oscillation is nearly sinusoidal and can be explained by the linear theory used by Helmholtz and all other early investigators. Simply stated this theory says that a feedback oscillation can be set up in which the oscillatory air flow through the reed creates an oscillatory pressure in the mouthpiece which causes an oscillatory motion of the reed which maintains the oscillatory flow through the reed. The incoming flow adds energy to the air column oscillation whose amplitude increases until the rate
of energy loss due to radiation and damping in the air column just balances the rate of energy input. Due to the phase relations required for energy input, the playing frequencies are just below the frequencies at which the air column input impedance, the ratio of mouthpiece pressure to air flow into the air column, is maximum. These maxima occur near the natural frequencies of a tube which is rigidly closed at the mouthpiece end, but are slightly less than these values because the reed acts as an elastic termination to the air column at frequencies far below the reed's own natural frequency.

Even at small amplitudes the linear theory is only an approximation to the actual case because the reed is a quite nonlinear valve, and at least weak harmonics of the playing frequency are generated in the air flow through the reed even when the reed motion is itself very nearly sinusoidal. When the blowing pressure, and hence the amplitude of oscillation, is increased so that the harmonic generation is significant, the linear theory breaks down since account must be taken of energy production and dissipation at the harmonics as well as at the playing frequency. It is this type of oscillation which is explained by the theory developed by Worman.

Worman \(^{32}\) used the fact that each of the components of the flow through the reed interacts independently with the air column, and thus each pressure component measured in the mouthpiece can be calculated by multiplying the amplitude of that flow component by the input impedance at the frequency of that component. This means that the mouthpiece pressure will be large for all harmonics at which the
input impedance is large. Each of these mouthpiece pressure com-
ponents contributes independently to the motion of the reed. At low
frequencies the reed responds equally to all frequencies, and the
reed motion contains the same relative amounts of all of the Fourier
components as does the mouthpiece pressure. However the reed
responds much more vigorously to pressure components that roughly
match the reed resonance frequency. Thus the reed motion is enhanced
at a particular component frequency either if the input impedance is
large at that frequency or if the reed resonance is near the fre-
quency of the component. In either case the larger reed motion
increases the corresponding component of air flow through the reed.
This flow further increases the pressure component which drives the
reed motion. The same type of feedback which sustains the oscillation
at the playing frequency can also occur at any harmonic and thus
increase the production of that component. In addition when one or
more of the higher components in addition to the fundamental is
sustained by this process, the oscillation becomes much more stable
in loudness, pitch, and tone color. The musician finds that the
note is "easy blowing" and has a very "clear" tone which is easily
distinguished through an ensemble or even an orchestra of other
instruments.

The enhancement of the overall tone quality produced by an air
column having several input impedance peaks near harmonics of the
playing frequency occurs in the low registers of all well constructed
woodwinds. The impedance peaks occur near the low order odd
harmonic components of cylindrical instruments such as clarinets and near all low order harmonic components of conical instruments such as oboes, bassoons, and saxophones; and it is these components whose amplitudes are increased by the feedback process. The oscillations in the upper registers of all reed instruments are stabilized by placing the reed resonance frequency near one of the components of the tone.

Because the properties of the input impedance at the upper components as well as at the fundamental are important in determining the quality of a low register tone, if the impedance peaks are nearly, but not exactly, aligned at harmonic intervals, the playing frequency of the note can change during a crescendo for the following reason. Consider a clarinet played near the note written $G_4$ for which the air column has two impedance peaks which are separated by nearly a three-to-one ratio in frequency similar to the case shown in Fig. 1. The fundamental component is supported by the lower frequency peak, and the third harmonic component can be enhanced by the second peak. On many clarinets these peaks are in fact separated by slightly less than a three-to-one frequency ratio. At very low playing levels where there is essentially no harmonic production, the playing frequency is entirely determined by the properties of the impedance at the fundamental component. The oscillation occurs at a particular frequency slightly below the first impedance peak as indicated by the upper arrows in Fig. 1. As the playing level is increased, the level of all of the higher components increases
relative to the fundamental. In particular the third component grows until it interacts significantly with the second impedance peak. Since the frequency of this peak lies a little below that of the third component of the pianissimo played note, the peak has less than its maximum possible effect on the oscillation. The net energy input and thus the stability of the oscillation could be increased if the playing frequency were slightly lowered. In moving the third component to increase its energy input, the fundamental will, of course, also be moved from its optimum position, even though the fundamental is moved only one third as far as the third component. The high amplitude playing frequency is indicated by the lower set of arrows in Fig. 1. In most musical cases the increased energy input at the third harmonic more than compensates for the slight decrease at the fundamental. Thus the playing frequency will fall slightly during a crescendo if the spacing of the impedance peaks is slightly less than a ratio of whole numbers. This effect is obviously undesirable and can be eliminated only if the peaks are accurately aligned at harmonic intervals. This explanation is implicit in Worman's work and has been confirmed by experimental work on real instruments done by Benade.

As previously mentioned, an oscillation which is well supported at several components has many special properties. The tone color is somewhat affected because the amplitudes of those components at which energy is directly produced by the feedback process at an impedance peak are significantly increased. However more important
to the musician is the increased stability caused by the additional feedback. These notes are judged to have a much "clearer" sound which is easily recognized by the listener even in the presence of other sounds, whether music or noise. The effect is even more pronounced when the listener is in a room and can receive information not only from the direct sound, but also from the reflected and reverberant sound. The study of these perceptual phenomena belongs to the field of psychoacoustics, and increasing attention to this study is paralleling the increasing ability of the physicist to provide the physical data on which such perceptions are based.

The stable musical oscillations supported by air column impedance peaks at several harmonics are special enough that it has been useful to define a term to describe them. Benade originally defined a "regime of oscillation" as

... that state of the collective motion of an air column in which a nonlinear excitation mechanism (the reed) collaborates with a set of air column modes to maintain a steady oscillation containing several harmonically related frequency components, each with its own definite amplitude.

As a result of the work described in this thesis we find that this definition needs to be modified in a manner to be described later in this section.

In applying perturbation theory to help align the air column impedance peaks to improve the stability of low register oscillations in clarinets, it was found that the properties of some clarion register notes were also affected. In retrospect this should have been quite puzzling because the notes in the second register do not
have an air column input impedance peak near any of their higher components. Reference to Fig. 1 shows that even the second harmonic component of the clarion register note (which is based on the second impedance peak) is far beyond the cutoff frequency of the instrument. There can thus be no cooperation of air column normal modes, and the explanation of the clarion register would seem to revert to the simple linear theory. However the linear theory is not adequate because it also cannot account for the changes in playing behavior of the clarion register notes with small changes in the bore. These effects are qualitatively the same in the upper register as in the lower register. The similarities in the playing behavior of all notes on the instrument postponed the realization that a separate explanation for the upper register notes was required. To some extent this delay was fortunate because it allowed the exploration of the consequences of intermode cooperation without unnecessary complication. However it also postponed the study of reed resonance effects, which are the subject of this work, much longer than would otherwise have been the case.

When the reed resonance is explicitly taken into account in the theory, the observed behavior of the upper registers of reed woodwinds can be explained. While the mathematics of the two cases is quite different, the same kind of behavior can occur near the reed resonance as occurs near air column input impedance peaks. For example it is possible to maintain a feedback oscillation at the reed frequency if the reed is only lightly damped. This can be done by
blowing into the mouthpiece with the teeth placed directly on the reed so that the lip does not add damping. The feedback mechanism which maintains these oscillations is basically the same as that for oscillations based on impedance peaks, but there are some minor differences. Because the reed frequency is far above the cutoff frequency of the air column, the input impedance is not high near the reed frequency, and thus the pressure variations in the mouthpiece are much smaller than they would be near an impedance peak. However these smaller pressure variations cause much larger reed motion near the reed resonance frequency than at any other frequency. If the reed damping is small enough, the small pressure variations can cause the reed to vibrate vigorously enough to allow the oscillation to be self sustaining. On the clarinet, oscillations of this type have frequencies generally above C7 and are not to be confused with the chirps and squawks produced by the inexperienced player which usually occur far below the reed frequency. Oscillations based on the reed frequency are musically useful in producing the extreme uppermost notes on most woodwinds. In addition this type of oscillation has proven useful as a research tool to locate the reed resonance frequency.

The reed resonance is musically important in another way. Just as a bore impedance peak near a harmonic of the playing frequency can stabilize the pitch and tone color of a note, placing the reed frequency at a harmonic of the playing frequency can bring about the same set of advantages. This is the mechanism which
stabilizes the upper register notes of all reed instruments. For example on a clarinet as the amplitude of a clarion register note is increased, the nonlinear flow control property of the reed produces some higher harmonics. Because the bore input impedance is not high at the upper components, the pressure amplitudes of these components do not grow as they would if an impedance peak were present. However the small pressure variations can have a large effect on the reed motion if the reed resonance frequency matches the frequency of one of the harmonics of the pressure spectrum. As was the case when the impedance peaks were properly aligned, the additional feedback at the harmonic which in this case matches the reed frequency minimizes the incidental frequency shifts and spurious noise, and stabilizes the attack and decay of the note. It is these physical attributes which the musician includes in his assessment of the superior musical quality of the note. The small change in the spectrum caused by the additional energy input at the reed frequency probably has only a minor effect on the tone quality.

Whatever the psychoacoustic reason, an oscillation whose main energy input is due to an air column input impedance peak at the fundamental and which receives additional support from the reed resonance at a harmonic of the spectrum is always considered to have good musical quality, and in fact has all of the special properties of a regime of oscillation based solely on impedance maxima. For this reason the original definition of a regime of oscillation should be changed to include those oscillations in which additional
energy input can come from a properly adjusted reed resonance
frequency, as well as from input impedance peaks. A more complete
definition follows:

A regime of oscillation is that state of the collective motion
of a nonlinearly excited oscillatory system in which the nonlinear
property of the excitation mechanism collaborates with a set of
the modes of the entire system (including any possible modes of
the excitation mechanism itself) to maintain a steady oscillation
containing several harmonically related frequency components,
each with its own definite amplitude and phase.

(The wording in this definition has intentionally been made general
enough to include oscillations in systems other than just reed
woodwinds.) While the high frequency oscillation whose fundamental
is near the reed frequency is not formally included in this defini-
tion, it is similar enough in musical quality that it will be called
the "reed regime." The definition now includes all normal musical
oscillations of reed instruments, although it does not include the
so-called multiphonics, most of whose components are inharmonically
related.

It will be shown in Sec. IV-B that the player can comfortably
adjust the reed resonance frequency over a fairly wide range by
changing his embouchure. Tight embouchure corresponds to high reed
frequency and loose embouchure to low reed frequency. Good musicians
do adjust the reed frequency to match a harmonic of the note they are
playing as a matter of course. They have learned for each note that
embouchure which gives best response and best tone quality, and they
hope that this "best playing" embouchure also gives the correct
pitch for the played note. This thesis outlines the physics which
underlies our understanding of the methods which can be used to properly adjust the notes in all registers of woodwind instruments to achieve acceptable playing behavior with proper playing pitch.

It is generally true on most reed instruments that in playing up the scale, the embouchure is consistently tightened so as to keep the reed frequency matched to the same serial number component for several successive notes in the scale. However when the reed frequency rises to a point where the player cannot comfortably raise it further, the embouchure is abruptly loosened, and the reed frequency is retuned to match a lower harmonic of the spectrum. One finds notes on all instruments which can be played with either one of two different embouchures which tune the reed frequency to different harmonics of the spectrum. For example, the clarinet is normally played so that the reed frequency matches the third harmonic of the playing frequency of the notes written from about C₅ (700 Hz) to about C₆♯ (990 Hz), the second harmonic of several notes above C₆, and the fundamental of the very topmost notes. C₆ (930 Hz) or D₆ (1050 Hz) can be played with the reed frequency matching either the second or third harmonic. The tendency to tighten the embouchure for the next highest note and loosen it for the next lowest note to keep the reed frequency adjusted when it is near the middle of its range would always be true if the air column showed no resonant behavior at all beyond its tone hole lattice cutoff frequency. However, as can be seen in Figs. 3-5, on real instruments the cutoff is not total, and there are always some reflections from the bell and from
irregularities in the bore. These cause small, irregularly spaced impedance peaks beyond cutoff. Any one of these peaks which happens to fall near the natural frequency of the reed can play a significant role in the tone production mechanism even though it is so small that it would have little effect in any other frequency range. By making small changes to the bore to move one of these small peaks to fall at the harmonic of the playing frequency at which the reed frequency will be set, the playing qualities of the note can be vastly improved. It was unplanned adjustments of this type which caused the previously unexplained change in playing behavior of the clarion register notes which was mentioned earlier.

As another illustration of the kind of phenomena which can now be explained by the theory which includes the properties of the reed resonance, it is common practice on some instruments to play the entire upper register without the use of a register hole. Bassoonists in the United States normally "flick" open a register hole to start the note and then leave it closed. The Baroque oboe has no register hole at all. These instruments are routinely played in the upper register despite the fact that a stable regime of oscillation is possible in the low register and would certainly be favored if the reed resonance did not help the upper register regime significantly. It is also the practice of some clarinet teachers to require their students to practice the clarion register without opening the register hole. Although no one would choose to play this way, it is possible to play the entire clarion register without
the register hole if the embouchure is always properly set. If the embouchure is not set properly the note will either drop to the low register or the instrument will not play at all. Practicing without using the register key is a very good way for a player to quickly improve his playing after not having played for some time. It can also help the well practiced player "warm up" before a performance. These things are all possible because of the extra energy input which takes place at the reed frequency when it matches the frequency of a component of the tone.
CHAPTER II

INITIAL ATTEMPTS AT A COMPUTER SOLUTION

At the start of this work it was hoped that it would be possible to develop a method of solving the coupled nonlinear equations developed by Worman which would be practical for solving more than the few simple cases which he considered. Worman\textsuperscript{34} developed a method of solving for the first three harmonics of a spectrum whose harmonic content was negligible beyond that point. Such a regime is the sort which is produced when playing on a tube whose tone hole lattice cutoff frequency is a little more than twice the frequency of the first air column input impedance peak. This assures that the impedance at all harmonics of the playing frequency is small. Worman further limited his consideration to those cases for which the reed resonance frequency is far above the frequency of the impedance peak so that the reed resonance effects described in this thesis could be ignored. This combination of parameters insures that the strengths of all of the upper harmonic components will be small compared to the component at the playing frequency, and thus that the three component approximation is reasonable. After using a computer to calculate the coefficients in his equations, Worman did all of the algebraic manipulations involved in the successive approximation solution by hand. In this way he was able to gain insight into which terms in the equations were growing and which were remaining small.
and could be neglected. This method was sufficient to show the validity of the theory, but it was not at all convenient for the study of the effects of changes in the parameters.

We had hoped that this same general method could be used for the present study but that the time involved in solving each case could be greatly reduced by allowing the computer to do more of the work. A set of programs was written to do all of the algebraic manipulations while leaving all of the decision making responsibilities involved in the successive approximation with the operator. With these programs the time required to obtain the three component spectrum for Worman's one resonance tube was reduced to a couple of hours. As Worman observed, simple iterative procedures were found to diverge. However by successively making small changes to each of the input pressure amplitudes to find their effect on the solution, it was possible to continuously improve the solution. The pressure spectrum calculated in this way was an excellent match to that of Fig. 20 of Worman's thesis.35

With high hopes the air column parameters in the programs were changed to approximate a more realistic musical system. To better model the low register of the saxophone or clarinet an additional impedance peak was placed near the frequency of the second or third spectral component; for the upper octaves the frequency scale was changed so that the reed resonance frequency was in the neighborhood of the second or third harmonic of the playing frequency. The computer programs which had been written did not provide an acceptable
solution for any of these cases because even when the Q of the second impedance peak or of the reed resonance was lower than normal in reed instruments, the amplitude of the second or third harmonic of the spectrum was large enough that it was not reasonable to neglect those components beyond the third. In addition the time required to find even a poor solution was greatly increased over the single resonance case. Each successive approximation took longer because the equations did not seem to have the same systematic behavior which they had previously had.

By introducing more spectral components into the computer programs, these problems could probably have been greatly reduced. However this extension would have been quite a major undertaking, and was not attempted at this time. In its present form the software system is divided into two sections, each of which takes nearly the entire 16k words of core of a PDP 9/L computer. To include more components in the solution, it would probably have been better to move to a larger computer. This would have required rewriting most of the existing software. In addition, after seeing the magnitude of the effort made by Schumacher\textsuperscript{36} to solve this same problem on a computer (although by a quite different method), I decided to search for another way to attack the problem. By rearranging and studying the equations themselves, a significant amount of information has been extracted, and this is presented in the subsequent chapters of this thesis. It might now be possible to develop a numerical method
of solution based on these new insights if such a solution were desired.
CHAPTER III

THE THEORY

The theoretical treatment to be presented here is similar to that developed by Worman; however a major emphasis will be placed on the reed characteristics which played only a peripheral role in the earlier work. In all cases the notation is chosen to match that used by Worman. The reader is warned that at times many pages of tedious algebra transpire between successive equations in the presentation. While these calculations are straightforward, many of them are so lengthy that the probability of completing them error-free would be very small were it not for the vast amount of structural order possessed by the equations.

Numerical values for all of the physical quantities used are listed in Table I. In nearly all cases these are the same values which were measured by Worman and used in his work. They also agree quite well with measurements carried out by others. In two cases, however, the data in Table I differ significantly from those used by Worman. In the present treatment the reed resonance frequency $\omega_r$ is treated as an adjustable parameter whose value can be set within certain limits by the player. The explicit determination of these limits is described in Sec. IV-B. In addition the reed effective area, defined in Eq. 6 has been changed to slightly less than half of Worman's value for the reasons explained in Appendix B.
A. Description of the System to be Studied

This work will deal specifically with an idealized clarinet-like system, although many of the results apply to all reed instruments for which the Bernoulli force on the reed tip (due to the air flow through the reed) can be neglected. Worman estimated the magnitude of the Bernoulli force for single reed instruments and found that in many cases it can be neglected. It is reasonable to postpone consideration of the Bernoulli force at this time because acceptable musical instruments can be made in which the force is negligible, although small changes in this force produced by altering the dimensions of the mouthpiece profile are readily perceived by the player and can have considerable musical significance. The present formulation does not accurately describe the double reed instruments because their reeds are strongly influenced by the Bernoulli force. However it is known that many generalizations from the present work do apply to them. The model clarinet system used in the theory is composed of a reed mounted at one end of a particular musical air column. A schematic representation of the model is shown in Fig. 2. The reed and air column are both assumed to behave as damped linear oscillators which are coupled because the mouthpiece pressure provides the driving force for the reed motion, while the air flow through the reed adds energy to the oscillations in the bore. This air flow, and thus the coupling it provides, is a highly nonlinear function of both the pressure difference across the reed and the reed tip opening. The reed thus plays a dual role in the model. It acts
as a linear oscillator at the end of the air column driven by the pressure variations in the mouthpiece, and it also serves as a nonlinear flow control valve which can add energy to the oscillation of the air column.

The assumption that the reed behaves as a linear oscillator introduces the restriction that the reed motion must not be so large that the reed beats against the tip of the mouthpiece. In practice, for clarinet reeds and mouthpiece designs normally used by orchestral players in the United States, such beating takes place only at loud playing levels. If we look a little more closely at the motion of a clarinet reed, we find that it may not behave as a linear oscillator even when it is not beating. The facing, that part of the mouthpiece over which the reed is mounted, is curved, and as the reed closes the aperture through which the air flows, it rolls down onto the facing thus shortening the effective vibrating length. As the reed opens it rolls away from the facing and effectively becomes longer. This type of nonlinearity is treated in Appendix A where it is shown that the resonance frequency of the reed is shifted by an amount which is second order in the coefficient of reed nonlinearity and is thus negligible. Perhaps more important, the nonlinearity causes a small amount of mainly second harmonic generation in the reed motion. The harmonics produced in this way will couple to the air column oscillations and thus influence to some extent the overall feedback mechanism. Explicit introduction into the theory of the nonlinearity of the reed dynamics on top of the nonlinear flow control character-
istic which will be included would vastly complicate the mathematical analysis without making any changes in the general behavior of the equations. Various coupling coefficients would be changed, but since the nonlinearity in the flow control characteristic of the reed is much larger than that in the reed response, these changes can be thought of as perturbations. Furthermore data taken by Backus of reed opening versus pressure difference across the reed shows that, at least far below its resonance frequency, the reed acts very much like a linear oscillator driven below resonance. For these reasons, the reed will be treated as a linear oscillator.

The oscillations of the air column are also assumed to be linear, and here the approximation is much easier to justify. The only major source of nonlinearity in the air column of musical instruments is turbulence at the sharp corners at the edges of tone holes and in the joints of the instruments. By carefully rounding these sharp corners, the turbulence effects can be minimized so that they only become important at loud playing levels where the other assumptions of the theory also fail.

If the reed is assumed to be a damped linear oscillator driven by the periodic pressure difference across it, the differential equation for the displacement of the reed tip, \( y \), is

\[
\frac{d^2y}{dt^2} + g_r \frac{dy}{dt} + \omega^2 r y = - \frac{1}{\mu_r} p
\]

(1)

where the reed is characterized by its resonance frequency \( \omega_r \), half-power bandwidth \( g_r \), and effective mass per unit area \( \mu_r \). \( p \) is
the pressure difference encountered in crossing from the outside to
the inside of the reed, and the negative sign occurs because a pos-
itive pressure difference tends to close the reed. Solving this
equation for the sinusoidal excitation \( p = A \text{e}^{i\omega t} \),

\[
y = - \frac{p}{\mu_r \left( \omega^2 - \omega_r^2 - i \omega \omega_r \right)} = - D(\omega) p
\]

\[
D(\omega) = d e^{i\delta}
\]

\[
d = \left\{ \mu \left[ \left( \omega^2 - \omega_r^2 \right)^2 + \omega^2 \omega_r^2 \right] \right\}^{\frac{1}{2}} - 1
\]

\[
\tan \delta = - \frac{\omega \omega_r}{\omega^2 - \omega_r^2}.
\]

D(\omega) is the complex reed response coefficient whose magnitude and
phase are \( d \) and \( \delta \) respectively.

The air column chosen for the theoretical study is similar to
that of a clarinet. The basic air column parameter which enters the
theory directly is the bore input impedance \( Z_b \)--the ratio of
acoustic pressure to volume flow at the tip of the mouthpiece where
air enters the air column. The incoming flow divides into two parts,
one entering the air column and the other filling or leaving the
space occupied by the reed as it swings back and forth. The pressure
which drives the flow into each of these regions is the pressure
within the mouthpiece. An impedance can be defined for each of these
parts of the flow, and since the pressure associated with each is the
same, the total input impedance of the air column \( Z \) is the "parallel"
combination of the input impedance of the air column \( Z_b \) and the
impedance associated with the reed \( Z_r \).
\[
\frac{1}{Z} = \frac{1}{Z_d} + \frac{1}{Z_r}
\] (4)

The input impedance of the bore is determined jointly by the dimensions of the closed-hole and open-hole sections of the bore. At low frequencies a wave originating in the mouthpiece is strongly reflected in the region of the first open hole. At frequencies for which these reflections return to the mouthpiece in phase with the exciting flow, the mouthpiece pressure amplitude, and hence the input impedance, becomes very large. These peaks are near the natural frequencies of a tube whose length is that of the closed-hole section and which is rigidly closed at the mouthpiece end. For frequencies above a cutoff frequency determined by the dimensions and layout of the open tone holes, reflection no longer occurs, and the wave is propagated into the open-hole section. The wave is eventually reflected at the lower end of the instrument, but since there is significant radiation from the open tone holes, the reflected wave has very small amplitude and thus the input impedance beyond the cutoff frequency is nearly constant. Typical input impedance curves for three notes of the clarinet are shown in Figs. 3–5. For simplicity, all calculations in this report are done assuming that beyond the cutoff frequency the input impedance of the bore is strictly constant and equal to the characteristic impedance of the tube. For comparison with these calculations, a clarinet-like system, described in Sec. IV-A was built with very flat impedance beyond cutoff. The next section of this chapter describes the musically important case in
which the small impedance peaks beyond cutoff can become a significant part of the oscillation mechanism if they fall near the reed resonance frequency where the reed transconductance is quite large.

The acoustic impedance associated with the flow into the region behind the reed is found by calculating this flow and dividing it into the mouthpiece pressure. The required volume flow is just the volume per unit time swept out by the reed as it swings back and forth. If \( w \) is the width of the reed, \( x \) is a coordinate which measures position along the reed from the reed tip, and \( Y(x) \) is the reed displacement from its equilibrium position at the point \( x \), then the acoustic volume flow associated with the reed motion is

\[
 u_r = \int_w \frac{dY}{dt} \, dx \tag{5}
\]

where the integration extends over the entire moving length of the reed. It is assumed that all points on any line perpendicular to the length are equidistant from the facing and move in phase. For sinusoidal excitation, this may be written

\[
 u_r = w \int \frac{dY}{dt} \, dx = S_r \frac{dy}{dt} \tag{6}
\]

where \( S_r \) is an effective reed area and \( y \) is the displacement of the reed tip from equilibrium. Combining Eqs. 2 and 6 to find \( u_r \), the reed impedance is found to be

\[
 Z_r = -\frac{P}{u_r} = -\frac{P}{S_r \frac{dy}{dt}} = \frac{\mu_r (\omega^2 - \omega_r^2 - i\omega g_r)}{-i\omega p S_r} = \frac{\mu_r}{\omega S_r} [\omega g_r + i(\omega^2 - \omega_r^2)] \tag{7}
\]

According to Eq. 4 this impedance is in parallel with the input impedance of the bore to yield the total input impedance of the air.
column. Fig. 6 shows a typical air column input impedance curve for a cylindrical instrument such as a clarinet. The dip in the total impedance at the reed natural frequency is caused by the decreased reed impedance in this frequency range. This impedance curve will be used in the theoretical discussion to follow.

B. Derivation and Discussion of the Equations

Describing the Oscillating System

The starting point of the derivation is Backus's expression for the acoustic volume flow through the reed aperture which, rewritten in terms of the present notation, is

\[ u = B p^{2/3} (y+H)^{4/3} \]  \hspace{1cm} (8)

where \( p \) is the pressure difference across the reed, \( H \) is the equilibrium opening of the reed tip, \( y \) is the reed tip displacement from equilibrium, and \( B \) is a dimensional constant whose value is

\[ 37 \text{ g}^{-2/3} \text{ cm}^{7/3} \text{ sec}^{1/3} \] in CGS units and \[ .08 \text{ kg}^{-2/3} \text{ cm}^{7/3} \text{ sec}^{1/3} \] in SI units. Backus's experiments were carried out under nonoscillatory conditions and thus the effects of the inertia of the air mass in the reed opening are neglected. However because the reactance due to this inertia is much smaller than the acoustic resistance implied by Eq. 8 at all frequencies of interest, this relation will be used without correction.

Eq. 8 is expanded in a two dimensional Taylor series about some appropriate values of \( p \) and \( y \), and coefficients of like powers of \( p \) and \( y \) are collected to yield
\[ u = \sum_{i,j=0}^{\infty} F_{ij} p^i y^j. \] (9)

The values of the \( F_{ij} \) used throughout this work appear in Table II. They were originally calculated by Worman. Next the relationships between pressure and flow of Eq. 10 and pressure and reed displacement of Eq. 2 are used to eliminate \( u \) and \( y \) from Eq. 9. The mouthpiece pressure is the product of the air flow into the air column and the input impedance \( Z \), whose magnitude and phase are \( z \) and \( \zeta \). The mouthpiece pressure is the difference between the blowing pressure \( P \) and the pressure difference across the reed \( p \). Thus

\[ (P-p) = Zu = ze^{-i \zeta} u. \] (10)

The reed response function \( D(\omega) \), which relates the reed displacement to the pressure difference across the reed, is defined in Eqs. 2 and 3. It must be remembered that \( D \) and \( Z \) are defined only for sinusoidal excitation, and thus for the musical case where the variables contain several harmonically related frequency components, Eqs. 2 and 10 must be considered as operator equations.

For a periodic oscillation each of the variables \( u, p, \) and \( y \) may be expanded in a Fourier series.

\[ p = \sum_{n=0}^{\infty} p_n \cos(n \omega t + \phi_n) \] (11a)

\[ u = \sum_{n=0}^{\infty} u_n \cos(n \omega t + \phi_n) = \frac{P}{z_0} - \sum_{n=0}^{\infty} \frac{p_n}{Z_n} \cos(n \omega t + \phi_n + \zeta_n) \] (11b)

\[ y = \sum_{n=0}^{\infty} y_n \cos(n \omega t + \chi_n) = \sum_{n=0}^{\infty} \frac{p_n}{n} d_n \cos(n \omega t + \phi_n + \delta_n) \] (11c)

Here \( n \omega \) is the \( n^{th} \) harmonic of the playing frequency \( \omega \), and the
subscript \( n \) signifies that the variable is to be evaluated at the frequency \( n\omega \). Eqs. 2 and 10 have been used to express \( u_n \) and \( y_n \) in terms of \( p_n \). Eqs. 9 and 11 can now be combined to yield a single equation for the amplitudes of the Fourier components of the pressure difference across the reed.

\[
\frac{P}{z_0} - \sum_{n=0}^{\infty} \frac{p_n}{z_n} \cos(n\omega t + \phi_n + \zeta_n) = 0
\]

\[
\sum_{i=j=0}^{\infty} p_i \left[ \sum_{n=0}^{\infty} \cos(n\omega t + \phi_n) \right] \left[ \sum_{n=0}^{\infty} p_n \cos(n\omega t + \phi_n + \delta_n) \right] = 0
\]

For musical oscillations it is always observed that the energy in the higher components is small beyond a certain point. Because of the increased radiation efficiency of the higher components, they are proportionally even smaller in the internal spectrum than outside the instrument. It is thus reasonable to terminate the Fourier series at some point. To study the general nature of the oscillations it is necessary to retain only the first few terms, and thus in this discussion the Fourier series will be terminated with \( n=3 \). Detailed numerical calculations would require keeping more terms. In the present treatment the Taylor series is terminated with \( i=j=2 \), although the effects of including higher order terms will be discussed in Sec. III-D. As with all series approximations, keeping only the first few terms is expected to give an acceptable approximation only at small excitation amplitudes. In this case, however, the second order approximation has been found to give at least qualitative agreement with experiment at all amplitudes for which the reed does
not beat against the tip of the mouthpiece. Agreement at such high amplitudes is unexpected, and a possible explanation for this behavior is given in Sec. III-D which discusses the effects of including higher order terms.

At this point the products indicated in Eq. 12 are expanded. Because of the linear independence of sines and cosines of different frequencies, the resulting equation can be divided into a set of coupled nonlinear equations each of which contains the coefficients of a single frequency sinusoid from Eq. 12. This set of equations can be written in the form which appears in Table III. While this same set of equations including terms to third order appeared as Worman's Table II, the present formulation makes it much easier to deduce the effects of changing the reed and air column parameters. The $A_i$, $B_i$, and $C_i$ in the present equations are to be considered as constants whose value depends on the reed parameters, the frequency, and the nonoscillatory pressure component $p_0$, but not on the amplitudes of the oscillatory components. At higher amplitudes it is necessary to consider terms beyond the second order, and in that case the $A_i$, $B_i$, and $C_i$ contain arbitrary powers of all of the $p_i$.

One notices that one possible solution to the equations of Table III is $p_i=0$, $i=1,2,3,...$. This nonoscillatory solution is possible for any value of the blowing pressure $P$. As $p_0$, the nonoscillatory pressure difference across the reed, is increased from zero, the only way for an oscillation to start is for the denominators of both of the expressions in Eq. B to be simultaneously zero.
\[
p_1 = \frac{a_1 p_0 p_2^+ a_1 p_0 p_3^+ a_2^+ p_2 p_3^+ \cdots}{\cos \frac{\zeta_1}{z_1} - A_1(d_1, \delta_1)} = \frac{b_0 p_0 p_2^+ b_1 p_0 p_3^+ b_2^+ p_2 p_3^+ \cdots}{\sin \frac{\zeta_1}{z_1} - A_2(d_1, \delta_1)}
\]  
(B)

Below the reed resonance for reed instruments \(A_1 > 0\) and \(A_2 < 0\), and the phase of the impedance at the playing frequency is restricted to the range

\[
-\frac{\pi}{2} < \zeta_1 < 0
\]  
(13)

Since \(A_1\) and \(A_2\) are different functions of \(p_0\), their relative magnitude can be adjusted to make both denominators zero and thus start the oscillation by changing \(p_0\). It is thus possible for both the mathematician and the musician to set the system into oscillation. With \(p_0\) properly set, both denominators will vanish if

\[
1 - z_1 (A_1^2 + A_2^2)^{1/2} = 1 - z_1 A = 0
\]  
(14)

which has the appearance of the standard criterion for a feedback oscillation. \(A\) is the reed transconductance which is essentially constant at low frequencies, but rises sharply near the reed resonance. Fig. 9 is a graph of \(A\) versus \(\omega\).

Eq. 14 is rigorously zero only at the threshold for oscillation. If \(z_1 A\) is less than unity, energy is lost from the system at a greater rate than it is added, and any oscillation which is present will eventually die out. If \(z_1 A\) is greater than unity, energy is added to the system at a greater rate than it is lost and the oscillation will at first grow exponentially. However as the amplitude increases, the nonlinearities produce increasing amounts of higher harmonic components. Eqs. C and D of Table III show that the
amplitude of the \( n \)th component grows as the \( n \)th power of the amplitude of the fundamental component. Since the energy in most of the higher components is simply radiated from the bell of the instrument, the oscillation amplitude is stabilized when the excess energy generated in the feedback process is radiated at the higher components. Thus the condition of Eq. 14 can be somewhat relaxed. Oscillation can take place provided that

\[
\frac{1}{A} \geq 1 \quad \text{(15a)}
\]

or

\[
\frac{1}{A} = Z_{\min}. \quad \text{(15b)}
\]

Fig. 7 shows a calculated graph of \( Z_{\min} \) and the typical impedance curve of Fig. 6 on the same axes. For this reed and air column, oscillation can take place in only two small frequency regions. The impedance near the first peak is larger than the minimum required for oscillation, and thus a stable oscillation having its fundamental frequency in this region can be produced. All except the very highest notes of reed instruments have their playing frequencies near an air column impedance peak in this manner. Another type of oscillation is possible above the cutoff frequency for the system of Fig. 7. Near the reed frequency \( A \) becomes large and thus \( Z_{\min} \) falls. If the reed damping is small enough, as is the case in Fig. 7, \( Z_{\min} \) may fall below the total impedance near the reed frequency and thus produce the "reed regime of oscillation." It is shown in Appendix B that the minimum value of the reed quality factor \( Q_r \) for the reed regime to occur is generally greater than the actual \( Q_r \) under normal
playing conditions. \( Q_r \) in Fig. 7 is somewhat higher than normal.) However a musician can establish the reed regime by placing his teeth directly on the reed to eliminate the damping caused by the lip. Blowing in this way with the proper "tooth force" on the reed can produce a squeal at about 2–3 kHz, and it is shown in Appendix B that this type of oscillation will always occur within a few per cent on the low side of the reed frequency.

The extreme uppermost notes of reed instruments are actually a combination of the two types of oscillation explained above. On the clarinet, for example, played in the region above about 1400 Hz (written \( G_b \)), the air column input impedance peaks are essentially obliterated by the presence of the tone hole lattice cutoff phenomenon and so are not large enough to sustain an oscillation in the usual way. It also turns out that the reed frequency cannot comfortably be lowered enough to place it at the desired playing frequency, and \( Q_r \) is not large enough to support a normal reed regime in any case. However if the reed frequency is lowered sufficiently, then the resonantly enhanced reed transconductance below the reed frequency can interact with the small impedance peak in the vicinity of cutoff to produce an oscillation which is a kind of hybrid of the two types discussed earlier.

The discussion has so far been primarily concerned with the dominant means of energy production which takes place at the playing frequency. It is also possible for energy to be added to the system at any of the harmonic components of the generated tone. We see from
Eqs. C and D of Table III that the denominators of the expressions for the amplitude of the \( n \)th component \( p_n \) vanish under the same conditions which cause the denominators of the expressions for \( p_1 \) to vanish. Thus the conditions on \( z_n \) and \( A(n\omega) \) to maximize the amplitude of the \( n \)th component are the same as the conditions on \( z_1 \) and \( A(\omega) \) to maximize the energy production at the playing frequency \( \omega \).

\( p_n \) is increased either by increasing \( z_n \), which can be done by moving an input impedance peak nearer to a harmonic of the playing frequency, or by increasing \( A(n\omega) \), which is done by moving the reed resonance frequency nearer to such a harmonic. In either case if \( z_n A(n\omega) = 1 \), then additional energy is added to the system at the \( n \)th component, and the constraint of the additional feedback loop makes the regime much more stable in amplitude, frequency, and harmonic content. Incidental FM and spurious noise are generally reduced, and the attack and decay transients are shortened and stabilized. Of course it is quite unlikely that the denominators of all of the expressions of Eqs. B, C, and D of Table III would rigorously vanish at the same \( \omega \) for any value of \( n \). The value of \( p_0 \) can be adjusted to "fine tune" the denominators of both expressions in Eq. B to zero, but in general that same value of \( p_0 \) would not make the denominators of Eqs. C and D also vanish. All that is really required to stabilize a regime of oscillation is that the denominator of \( p_n \) be small when the denominator of \( p_1 \) vanishes.

The adjustment of the positions of the impedance peaks so that one or more of the higher mode peaks fall at some harmonic of
the playing frequency must be done by the instrument maker. Small perturbations to the bore or the tone hole sizes are sufficient if the instrument has been properly designed. On the other hand, the adjustment of the reed resonance frequency to be at a harmonic of the playing frequency must be accomplished by the player. It will be shown in Sec. IV-B that the clarinet player can comfortably adjust his embouchure to place the natural frequency of the reed between about 2 kHz and 3 kHz. Players of other instruments can cover an analogous range suited to the pitch level of their particular instrument. We find that the clarinet player can match the reed frequency to the third harmonic of the notes from about 700 Hz (written $G_5$) to about 1000 Hz (written $C_6^\#$), and the second harmonic of the notes to about 1400 Hz (written $G_6$). Beyond this point the oscillations are all the hybrid type of oscillation for which the reed frequency is not far from the playing frequency. This harmonic matching is a part of the craft of the musician, and the good player does it routinely and unconsciously.

C. The Role of the Phase of the Impedance

The previous section dealt with the requirements on the magnitude of the air column input impedance for oscillation to take place. This section deals with the phase requirements. For this discussion it will be convenient to define the phase of the transconductance as

$$\tan \alpha = \frac{A_2}{A_1}. \quad (16)$$

Reference to Table III shows that the requirement for both
denominators in Eq. B to vanish is
\[
\frac{\sin \xi_1}{\cos \xi_1} = \tan \xi_1 = \frac{A_2}{A_1} = \tan \alpha
\]  
(17a)

or
\[
\alpha = \xi_1.
\]  
(17b)

As shown in Fig. 11 for a basically cylindrical tube, Eq. 17b holds in two frequency regions. The first is just below the first impedance peak, and the second is just below the reed frequency. These are the same frequency regions in which the magnitude of the impedance is large enough to support an oscillation. Thus for this particular impedance curve, both the magnitude and phase requirements are satisfied at either of the two frequencies.

The restrictions on \(\xi_n\) to minimize the denominators of the expressions for \(p_n\) are similar. Since the air column peaks near the higher harmonic components are still far below the reed's own natural frequency, Eq. 17b remains valid. However it is not required that the denominator of the expression for \(p_n\) vanish, but only that it be small. Thus we must have
\[
\xi_n \approx \alpha
\]  
(18a)

or
\[
|\xi_n| \ll 1,
\]  
(18b)

but there is no restriction on the sign of \(\xi_n\). At frequencies near that of an input impedance peak, the second term in the denominators of Eqs. C and D is small regardless of the sign of \(\xi_n\). Since the first term depends only on \(\cos \xi_n\), it is also insensitive to the
sign of $\zeta_n$. Thus even though $\zeta_n$ changes sign at the impedance peak, the behavior of $p_n$ is very much the same whether the frequency of the peak is slightly above or slightly below the frequency of the $n$th component. $p_n$ will be largest when the peak is properly located to minimize the denominator, but the behavior is nearly symmetric as the peak is moved in either direction from this frequency. This is in sharp contrast to the behavior at the fundamental frequency of the oscillation where it is strictly required that the playing frequency fall at a particular frequency slightly below the frequency of the impedance peak at threshold.

Looking again at Fig. 11, we find that the curves of $\alpha$ and $\zeta$ do not diverge symmetrically above and below their point of intersection near the reed frequency. The curves diverge more quickly above than below $\omega_r$. Thus the denominators of Eqs. C and D of Table III will grow faster as the reed frequency is moved below its optimum position than when it is moved above that position. This means that when the reed frequency is placed near some harmonic of the playing frequency in a regime of oscillation, the tone quality of the note will decrease faster as the reed frequency is lowered than as it is raised. However it is true that the denominator passes through a local minimum at the position of maximum energy input, and small changes from this position increase the denominator from zero only quadratically. Thus for small changes, the behavior should be symmetric as the reed frequency is either raised or lowered. In any case the denominators will grow slowly at first for small changes in
\( \omega_r \), and since it is not required that the denominators be zero, but only that they be small, an oscillation can be somewhat stabilized whether the reed frequency is slightly above or slightly below its optimum position. Again this differs from the case of the reed regime where the playing frequency is near the reed frequency, and the denominator of Eq. 8 must actually vanish at the playing frequency. In that case the restrictions on the phase are quite strict.

D. The Effects of Including Higher Order Terms

When terms beyond the second order are included in the Taylor series expansion of Eq. 9, the \( A_i \), \( B_i \), and \( C_i \) in the equations in Table III are not independent of the amplitude of oscillation. In this case all of the "constants" become nonlinear functions of all of the component pressure amplitudes. In fact the \( A_i \) become different functions in different equations. Rough calculations show that none of the values of these variables will be changed by more than a few per cent even if the amplitudes of some of the higher components are as large as ten per cent of the amplitude of the component at the playing frequency. Effects of this magnitude would certainly have to be included if detailed calculations were to be attempted, although it is probably not necessary to include terms beyond the third order in the Taylor series expansion to get solutions correct to about one per cent. It would be necessary to include more than the first three components at that level. The fact that the \( A_i \) are different functions in different denominators means that the conditions for maximizing the \( n^{th} \) component amplitude are not the same as those
for maximizing energy production at the playing frequency. However many of the higher order terms are the same from one denominator to another, and in any case their inclusion does not change the basic behavior of the solutions which has been presented in this chapter. It will still be true that there is a particular setting of the reed frequency or the frequency of an air column input impedance peak which maximizes the $n^{th}$ component amplitude and stabilizes the feedback mechanism, and that this setting is near to a harmonic of the playing frequency. The fact that the exact setting may be slightly changed by including higher order terms is of minor importance.
CHAPTER IV

EXPERIMENTS

A major difficulty arises in trying to devise experiments to confirm the theory developed in the preceding chapter. It is not really possible to measure the reed resonance frequency under actual playing conditions. The players lip provides sufficient damping that the system will not oscillate near the reed frequency in the reed regime as explained in Appendix B. Standard resonance measurement techniques cannot be used because the player cannot hold his embouchure steady long enough to allow the reed to be driven through its resonance frequency while observing the amplitude response. Experiments performed on a blowing machine cannot be considered to have taken place under actual playing conditions because the artificial embouchure in the blowing machine does not provide a good approximation to the conditions in the human mouth. While each of the experiments to be described in this chapter can be explained by the theory, no single experiment alone is sufficient to confirm the validity of the theory. All together, however, the series of experiments provides ample evidence as to its essential validity.
A. Description of the Experimental Clarinet-Like System

In order to correlate the experimental data with the predictions of the theory, it was desirable to have a clarinet-like air column whose input impedance is constant beyond the cutoff frequency. Theoretically this can be accomplished by fitting the end of the closed hole section of the air column with an infinite row of regularly spaced open tone holes. As a wave travels down the tube from the mouthpiece, it will be reflected in the region of the junction between the closed and open hole sections if its frequency is less than a cutoff frequency \( f_c \) given by

\[
 f_c = \frac{v}{2\sqrt{2\pi}} \frac{b}{a} (se)^{3/2}
\]

(19)

where \( a \) and \( b \) are the radii of the bore and open tone holes respectively, \( 2s \) is the interhole spacing, \( te \) is the effective wall thickness including end correction, and \( v \) is the speed of sound in the bore. Eq. 19 is actually an approximation which is valid at low frequencies where all of the distances \( (a, b, s, te) \) are much less than the wavelength. The general formula is available in the literature\(^6\). Above this frequency the wave is propagated into the open holes section with no reflection. A portion of this wave is radiated from each open hole, and while the radiation from each hole may be small, if the row of holes is infinite there is never any reflection and the input impedance of the air column is constant above cutoff. Of course no real system can be infinite. If the tube is simply terminated with an open end there will be reflections
from the end producing small impedance peaks whose spacing corresponds approximately to that of a tube without holes whose length is the total length of the open and closed hole sections of the system under consideration. The longer the tube the smaller these peaks will be due to the increased radiation damping from the tone holes. A system which is long enough to produce a sufficiently flat impedance beyond cutoff would be too long to be practical. However if a material with high damping and wave impedance equal to that of free space is placed in the tube, the wave will be heavily damped and not reflected by the damping material. Loosely packed Fiberglas wool fits this description nicely and was used in the tube which was constructed. If the damping section is sufficiently long the wave will be quite small by the time it is reflected back to the mouthpiece, and thus the resonant behavior beyond cutoff can be reduced as much as desired. The Fiberglas cannot start for some distance beyond the junction since frequencies below cutoff do propagate a short distance into the open hole section. However the reflected wave pressure amplitude is always negligible beyond the first several open tone holes and damping material beyond this point has essentially no effect on the low frequency behavior. Figs. 13 and 16 show the effect of adding the Fiberglas wool used as damping in the tube which was constructed.

In addition to a constant impedance beyond the cutoff frequency, it was desired to design a tube whose cutoff frequency could easily be changed. This feature was not used directly in the present
work but will be of use in the future. To this end, three identical rows of tone holes were placed symmetrically about the axis of the tube. Several open holes at the same position along the tube act acoustically similar to a single hole whose area is the sum of the areas of the individual small holes, as long as the holes are not so small that viscous forces at the walls dominate. One or two of these rows can be closed off with tape to achieve a total of three different cutoff frequencies. In addition, in any of these configurations, alternate holes can be covered to lower the cutoff frequency. Since the cutoff frequency varies as the square root of the area of the open holes, successively opening one, two, and three sets of holes should change the cutoff frequency approximately in the ratios of \(1: \sqrt{2}: \sqrt{3}\). (The ratios are slightly greater than this because the closed holes at each lattice position add slightly to the effective length of each section.) The tube was designed to have the second of its possible cutoff frequencies at 1500 Hz, the cutoff frequency of the \(B^b\) clarinet. It was used in this configuration in all of the experiments described in this chapter.

Fig. 12 is a drawing of the tube which is marked with identification number ST1. For reference, this tube will be called THL (tone hole lattice) followed by the number which is the approximate cutoff frequency when the distinction is necessary. Figs. 15-17 show the measured input impedance of THL in its three configurations when used as the termination for a long piece of clarinet-like tubing with no tone holes. As seen in these figures, the input impedance
does not change by more than about ±15 per cent beyond the cutoff frequency.

In order to be able to study several notes in an approximately chromatic scale, a second tube, analogous to a clarinet top joint, was built with the necessary tone hole spacing. To maintain the flat impedance beyond cutoff, the tone hole size was adjusted so that this top joint had the same cutoff frequency as THL1500. This was accomplished by using a single row of holes of the same size, thus halving the total hole area, while also halving the interhole spacing. A single set of two holes with the larger spacing was necessary at the lower end to allow for a normal tenon and socket joint to mate with THL. Fig. 18 is a drawing of this top joint which was given the identification number ST2. Figs. 18-20 show the measured input impedance curves for the system consisting of a standard cylindrical B♭ clarinet barrel and the tubes ST2 and ST1 with 0, 4, and 7 holes open on the top joint ST2. It will again be noticed that the input impedance of this system is flat within about 15 per cent beyond the cutoff frequency.

B. Determination of the Range of the Natural Frequency of the Reed

As mentioned before, the player has some control over the natural frequency of the reed. By tightening and loosening his embouchure he can change the playing frequency of notes in the clarion register by ±0.6 per cent (±10 cents) quite easily. It is
shown in Appendix III that this corresponds to a change in the
natural frequency of the reed of about ±15 per cent (250 cents). To
find the actual range over which the reed frequency can be adjusted,
the apparatus shown in the block diagram of Fig. 22 is used. The
reed frequency was approximately determined by measuring the fre-
quency of the reed regime oscillation while playing with a normal
embouchure. Because the reed $Q_r$ under normal playing conditions is
too small to support a reed regime oscillation, the additional
feedback through the electronic system was provided to allow the
oscillation to be self-sustaining.

The operation of this additional feedback loop is as follows.
A strain gauge mounted on the back of the reed is used as one leg
in a voltage divider. As the reed undergoes its periodic motion,
the voltage across the strain gauge acquires a small AC component
proportional to the reed curvature at the gauge. This signal is
amplified and fed to a speaker at the lower end of the clarinet-
like tube. Sound waves from the speaker can cause the reed to
vibrate thus providing the feedback necessary to set the system
into self-sustained oscillation if the phase shift in the feedback
loop is just right. The preferred oscillation frequencies are near
that of the reed resonance and those of the standing wave resonances
of the air column. The air column resonances are heavily damped by
placing Fiberglas wool in the tube, and the additional feedback is
reduced at low frequencies by band limiting the amplifier. Thus
it is ensured that oscillation can only occur near the reed resonance
frequency.

As mentioned before, with the amplifier turned off the system will not oscillate. If the amplifier amplitude response is flat and the phase response is tailored so that the phase shift in the feedback loop (including the shift due to the wave travel time up the tube) is a multiple of \(2\pi\) radians at all frequencies, then turning up the amplifier gain would be equivalent to increasing the characteristic impedance of the tube. In both cases the pressure variation caused by a particular flow through the reed would be increased. As this input impedance is increased by increasing the amplifier gain, the system will eventually go into oscillation near the reed frequency where the impedance required for oscillation is minimum.

However because in this type of oscillation the reed is driven near its resonance frequency, its response is nearly sinusoidal. Thus the feedback signal contains only a single frequency and the amplifier phase compensation can be considerably simplified. Since the phase shift needs to be accurate only at the oscillation frequency and not at its harmonics, a simple adjustable all pass filter can be used to adjust the phase shift in the feedback loop. If the phase is adjusted so that the oscillation takes place with minimum amplifier gain, then the oscillation again takes place as if the characteristic impedance of the tube had been increased to the point of oscillation. Its frequency is within a few per cent below the reed frequency as shown in Appendix B.
By playing with the electronics properly adjusted and with an embouchure characteristic of that used in normal clarinet playing, it is possible to set the oscillation frequency anywhere between about 2 kHz and 3 kHz. With somewhat more extreme changes of embouchure the frequency can be lowered to about 1800 Hz and raised to about 3400 Hz. It is shown in Appendix B that the reed resonance frequency will always be within about 10 per cent of the playing frequency when playing in the reed regime. This reed range seems reasonable in light of the results obtained in other experiments. Presumably the endpoints of the range would change slightly for different reeds and different mouthpiece facing designs.

C. Experiments on a Clarinet Blowing Machine

The theory of Sec. III-B predicts that second register oscillations should be stabilized if the reed frequency is set near a harmonic of the playing frequency. This section describes an experiment on a clarinet blowing machine which confirms that prediction.

The blowing machine consists of a rectangular cavity in which the mouthpiece is mounted. Clarinet-like upper joints can be attached to the outside of the cavity to create a normal musical air column. The dimensions of the cylindrical tube connecting the mouthpiece and upper joint are typical of clarinet barrels. The cavity surrounding the mouthpiece is connected through a length of 7/8 in. copper tubing to the outlet of a reversed vacuum cleaner to provide the blowing pressure to the cavity. A brass "tooth" covered with
silicone rubber "lip" presses the reed against the mouthpiece facing simulating normal blowing conditions. The position of the "lip" and the force applied by the "tooth" can be varied with adjusting screws. The blowing pressure is adjusted by changing the line voltage to the vacuum cleaner with a Variac. After considerable practice it was possible to adjust the blowing pressure, "tooth" position, and "lip" force to set the system into oscillation at any note on the clarinet. The silicone rubber material used as the artificial lip provided significantly less damping than the human lip. Thus the reed Q, in this experiment is higher than under actual playing conditions, and in fact is high enough to allow the reed regime to be produced as explained in Appendix B. The reed regime oscillation was again used to approximately determine the reed frequency. The conditions in the blowing machine are not a good reproduction of the conditions in the human mouth, and the sounds produced by the system are not high quality musical tones. However the effects of changing the artificial embouchure are the same as those of changing the actual embouchure when playing. The major difference between the conditions in the blowing machine and those under actual playing conditions is the fine control of the reed damping which the player can exercise by changing his lip tension and placement. This difference is sufficient to explain the observed behavior of the system in the blowing machine, and is unimportant for the present consideration.

The air column used was that described in the first section
of this chapter. Since this system does not have a speaker key or register hole like a normal clarinet, it was quite difficult to adjust the system to play in the upper register. The "tooth" position and "lip" force adjustments were critical. Both the low register note and the reed regime were much easier to obtain. However when the adjustments were properly made the clarion register note could be played, and further delicate adjustments brought something approximating musical tone to the oscillation. With these adjustments made, a little damping material such as Fiberglas wool or a handkerchief was placed lightly just outside the first few open tone holes. This provided enough damping to lower the bore impedance peaks below the threshold for oscillation. The system would then jump to the reed regime and in all cases where the clarion register oscillation had been stable the reed regime frequency was within 3 per cent (50 cents) of the second or third harmonic of the note in the clarion register which was played when the embouchure was set. For this experiment the clarion register regime was considered to be stable if the oscillation returned to the clarion register note when the damping was removed from the outside of the tone holes. The actual experimental results appear in Table IV.

Thus at least on the blowing machine and with an instrument whose impedance is flat beyond cutoff, in order to play with the best musical tone, the reed frequency should be set near a harmonic of the note being played and almost precisely at the frequency which maximizes energy production near the reed frequency.
The few cents discrepancy between the harmonic of the clarion register note and the playing frequency of the reed regime is understood as follows: the clarion register plays at the frequency which maximizes the total energy production at both the playing frequency and the reed frequency, while the reed regime maximizes energy input at one frequency only. The additional constraint on the clarion register regime explains the observed small differences from exact harmonic relationship of the clarion regime and the reed regime.

D. Effects of Reed Resonance on Spectrum and Tone Quality

It has been stated several times that if the reed frequency is placed just above a harmonic of the playing frequency, then that harmonic amplitude will be increased. It was also shown in the previous section that this setting of the reed frequency produces the best musical tone quality for clarion register notes on the clarinet. These two statements were reaffirmed in the following manner. A PZT ceramic microphone whose response is flat within about ±0.2 dB over the range of frequencies considered was mounted along the side wall of a mouthpiece to measure the mouthpiece pressure spectrum. This mouthpiece could be used on either the experimental clarinet system described in Sec. IV-A or on an actual clarinet. The clarinet was played and the mouthpiece pressure spectrum analyzed in two different ways.
The first method of analysis used a Nelson-Ross Model 001 spectrum analyzer oscilloscope plug-in. This unit provides an oscillographic sweep of the spectrum several times a second, so that the player can immediately see the spectrum and watch its development as he changes his embouchure. Fig. 23 is representative of the spectra obtained in this way. These spectra show that when any note is played with best musical quality, the amplitude of the spectral component which falls within the reed frequency range is always several dB larger than the amplitudes of its immediate neighbors. As the embouchure is changed, this component's amplitude falls quickly to be about the same amplitude as its neighbors. If the embouchure is significantly loosened to lower the reed frequency, the amplitude of the next lowest spectral component sometimes increases dramatically. Likewise if the reed frequency is significantly raised, the amplitude of the next highest component may be increased. These observations, while qualitative in nature, were made both by the author and by Arthur Benade several times over a period of several months on different instruments with consistent results.

In order to document these results in more detail, a second method of analysis was done. Using the experimental clarinet system, the mouthpiece pressure signal was recorded on a magnetic tape loop and played back through a GR 1900-A wave analyzer with its bandwidth set at 50 Hz. This is wide enough to span any small frequency shifts of any component of interest. Each note was
recorded and analyzed three times—once at the frequency which gave
the best tone quality from the musician's point of view, and once
each at frequencies 0.6 per cent (10 cents) sharp and 0.6 per cent
(10 cents) flat from that producing best quality. It is shown in
Appendix C that this corresponds to a reed frequency shift of about
±15 per cent (±2½ semitones). The spectra obtained from the record-
ings are shown in Figs. 23-25. It is clear that when the embouchure
is adjusted for best tone quality the component at the reed frequency
is maximized, and as the reed frequency is changed, sometimes the
next component in the direction of the change is increased.

The results of these experiments under actual playing condi-
tions can be explained as follows. One observes, as indicated in
the last section, that the best musical quality occurs when the reed
frequency matches a harmonic of the playing frequency. When this is
true, the theory of Sec. III-B predicts that the component at the
reed frequency is maximized. As the reed frequency is shifted,
the amplitude of that component which had previously matched the reed
frequency decreases drastically. If the reed frequency is moved far
enough that it comes near to another component, then this harmonic
amplitude increases for the same reason. Thus the results are con-
sistent with the conclusions of the last section and agree with the
predictions of the theory.

IMPORTANT NOTE!

One should not assume that these spectral changes alone are
the reason for judging a note to have good musical tone quality. To
the contrary, there are indications from practical music that the improvement is associated to a greater degree with the fact that when the reed frequency matches a harmonic of the playing frequency, the oscillation is stabilized by the increased feedback at the reed frequency. The incidental small frequency changes and the spurious noise present in the tone are thereby decreased. A proper study of such matters lies within the field of psychoacoustics rather than physics, and so lies outside the scope of the present inquiry.

E. Musicians' Experiments

Since the recognition of the importance of reed resonance effects, several attempts have been made by Arthur Benade to use this information to better understand and improve the playing quality of actual musical instruments. In all cases these experiments have given qualitative confirmation to the theory developed in Chapter III. The first of these involved placing the proper size blob of wax in the bell of a modern conservatory oboe. This rearranged the small impedance peaks and dips beyond the cutoff frequency in such a way that the playing qualities of some of the upper notes, which had previously been the worst on the instrument, were greatly improved. This occurred because a small impedance peak beyond cutoff was made to coincide with a harmonic of the playing frequency when the reed resonance frequency was also near that harmonic component. However for the oboe, the shape of the bell is important in determining the properties of the impedance below
cutoff. Thus the tuning of several of the notes in the low register was affected in ways which would have required major surgery to correct. The method was not usable for this particular instrument.

On a different conservatory oboe whose impedance peaks are not well aligned at harmonic intervals, Benade has found that on most notes there are several distinct embouchures which achieve "best" musical tone for this instrument. By paying close attention to what he was doing with his embouchure, he has been able to find the origins of these "best-playing" embouchures. Care must be taken in analyzing such experiments, however, because changes in the embouchure change not only the reed frequency but also the effective volume of the reed cavity which for conical instruments will change the position and spacing of the impedance peaks. The different "best-playing" embouchures occur when various sets of air column impedance peaks are aligned at harmonic intervals with each other or with the reed frequency. These cases can be distinguished from each other in ways such as the following. When the embouchure is adjusted so that the playing frequency of the second register note is exactly an octave above the playing frequency of the low register note, then the first two impedance peaks are accurately aligned; this is one of the "best-playing" embouchures for the low register. Another occurs for the low register note when the reed frequency is at an exact harmonic of the second impedance peak. This embouchure can be identified because it is a "best-playing" embouchure for both the first and second registers. Many other examples
of this type of behavior have been identified and explained. Of course an instrument with such multi-optimum behavior is not musically useful. The playing behavior of the entire instrument can be significantly improved if for each note the impedance maxima could be properly aligned with that embouchure which also placed the reed frequency at a harmonic of the playing frequency.

There is an exception to this general rule for the case of the Baroque oboe. This oboe has no register hole, and octave changes are made simply by changing the embouchure. In order to accomplish this consistently and unambiguously, the embouchures for the two octaves must be different, and each must provide a set of cooperations to stabilize the oscillation. In order to avoid unwanted octave shifts, the first and second impedance peaks should not be aligned with each other when the embouchure is set for either octave. This makes the reed frequency adjustment even more important because for many notes it is the only mechanism for additional energy input.

Two examples of clarinet behavior which can now be understood were presented earlier. It was stated near the end of Sec. I-B that the upper register of the clarinet can be played without opening the register hole if the reed resonance is always properly adjusted for maximum energy input. The hybrid oscillation in which a very small impedance peak interacts with the resonantly enhanced reed transconductance below the reed frequency was discussed in Sec. III-B.

As a final example of the application of the ideas presented
in this thesis, saxophone mouthpiece facing designs prevalent in the 1920's were such that the reed frequency could not be raised much above the playing frequency of notes in the top of the second register. The notes written at about D₆ could be achieved as reed regimes, but it was not possible to play many notes in the third register of the instrument. It was also not possible to play the second register without opening the register hole, because the reed frequency was too low to add energy to the oscillation at a higher component. More recent mouthpiece facing designs have allowed the reed frequency to be raised to a range analogous to that of the clarinet so that the third register is possible and the second register can be played without the register hole.
CHAPTER V

CONCLUSIONS

A. Problems with Extending the Present Formulation

This thesis has extended the work of Worman and Benade to show that the nonlinear flow control property of the reed couples the reed resonance into the oscillatory energy production mechanism when the reed natural frequency is near to a low order harmonic of the playing frequency. In this way the reed resonance can serve the same function as an input impedance peak in stabilizing an oscillation. The mathematics of the two cases is somewhat different since the reed resonance affects both the input impedance $Z$ and the reed transconductance $A$, whereas an input impedance does not affect $A$. The musical significance of the two cases is, however, so similar that it proves desirable to change the formal definition of a regime of oscillation to include those cases in which the additional energy input arises from a properly adjusted reed resonance.

Because of the analytical nature of Worman's method of solution, a number of simplifications have been made to the physical system to allow the major phenomena to be studied with a tractable mathematical formulation. However a number of dynamically and musically important effects have been neglected which should be investigated in future studies. There are two major effects in this
category. One is the Bernoulli force on the reed tip produced by the air flow through the reed. The Bernoulli force is very important in the double reeds and in single reed instruments whose mouthpiece design includes a high baffle at the mouthpiece tip. To include this force, the additional driving term

\[- \frac{1}{2} \frac{\rho u^2}{\mu \omega^2 (y+\delta)^2}\]

must be added to the right side of Eq. 1. The second major phenomenon which should be investigated is the large amplitude behavior when the reed beats against the tip of the mouthpiece. This again is especially important for double reeds which often beat even at fairly low oscillation amplitude. The beating reed might be treated in Worman's formulation by adding a position dependent force to Eq. 1. This might be approximated by the term

\[1 - e^{-a(y+\delta)}\]

where \(a\) is a large positive constant. In treating either the Bernoulli force or the beating reed, the direct solution of Eq. 1 becomes very messy if, in fact, it can be solved at all. In any case the mathematics would be vastly complicated, and it is not obvious that any useful information would be gained in the process.

Some information about the effect of the Bernoulli force can be learned just from observing the nature of the additional driving force term. The force is basically quadratic in nature, and thus its major effect is to more strongly couple adjacent frequency components. This can increase the overall feedback if one of the
components which is increased by the Bernoulli force is also near an impedance peak or the reed resonance frequency.

The difficulties encountered in extending the present formulation to include the effects which have been neglected in this thesis lead to the exploration for another method of solution which will allow the investigation of these important effects. Such a method is outlined in the following section. The proposed procedure has the additional advantage that it should be able to predict the transient as well as the steady state behavior of the musical system.

B. Possibilities for a New Method of Solution: The Time Domain

The method of solution discussed so far in this thesis is a frequency domain solution; i.e. the response of the system to a sinusoidal excitation is calculated as a function of frequency, and the resulting responses added in the proper way to find the total response to a complex excitation. The problems arise because the equation determining the flow through the reed (Eq. 8) is actually a time domain equation; it describes the flow at a particular time as a function of pressure difference across the reed and reed tip opening at that same time. It is thus necessary to transform the frequency domain representations of u, p, and y to their time domain counterparts (calculate their waveforms by forming a Fourier series) in order to proceed with the solution. This leads to the unwieldy series expansions which are a major problem. The difficulty can be eliminated if the entire problem is done in the time domain. The
time domain formulation results in a nonlinear integral equation which probably cannot be solved in closed form. However numerical methods can be used to obtain a solution.

The proposed new formulation may be quite similar to that used by McIntyre and Woodhouse in analyzing the oscillations of a bowed string. However I have not seen a formal presentation of their method, and there may be some significant differences. To the extent that the two methods are similar, it is encouraging, because the initial results of McIntyre and Woodhouse have been very promising. The proposed method is not especially similar to that used by Schumacher, even though both formulations result in integral equations. Schumacher's equation is essentially an eigenvalue equation with the steady state mouthpiece pressure waveform as the eigenfunction. He uses iterative methods to solve for the eigenfunction, but must start with both the playing frequency and a reasonable approximate initial waveform. The method is only valid for the steady state behavior.

In first setting up the new equation, the restriction inherent in Worman's formulation will be retained. The next sections will deal with methods to treat the Bernoulli force and the beating reed. In setting up the time domain solution, it is convenient to define a new set of variables. We will let \( p_m \) be the acoustic pressure in the mouthpiece which is the sum of the pressure due to the incoming flow through the reed \( p_L \) and the pressure of the wave reflected from the bottom of the instrument \( p_r \).
\[ p_m = p_i + p_r \]  

(20)

The air column and the reed will continue to be considered as linear systems which have impulse responses to pressure excitation \( g(t) \) and \( h(t) \). The impulse response of the air column can be calculated as the Fourier transform of its frequency response,

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{Z(\omega)}{Z_0} e^{i\omega t} d\omega, \tag{21} \]

where \( Z(\omega) \) is the parallel combination of the bore impedance and the reed impedance, and \( Z_0 \) is the characteristic impedance of the tube. The impulse response of the reed is not simply the Fourier transform of its frequency response. When the system is excited by an impulsive pressure, the reed is driven not only by the pressure impulse but also by the air column impulse response. Thus \( h(t) \) is given by

\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} D(\omega) e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega D(\omega) \int_{0}^{t} g(t-t') e^{i\omega t'} dt'. \tag{22} \]

The reflected pressure wave is calculated as the convolution of the incident pressure with the tube impulse response.

\[ p_r(t) = \int_{0}^{t} p_i(t') g(t-t') dt' \tag{23} \]

The reed displacement is the convolution of the driving force with the reed impulse response.

\[ y(t) = \int_{0}^{t} [P - p_m(t')] h(t-t') dt' \tag{24} \]

In both cases it is assumed that the excitation does not start until \( t=0 \). The driving force on the reed is the pressure difference across the reed, \( P - p_m \), where \( P \) is the blowing pressure in the mouth.

In terms of the present notation, Backus's expression for the
flow through the reed aperture is

\[ u = B(p - p_m)^{2/3} (y + H)^{4/3}. \]  \hspace{1cm} (25)

In the new formulation, \( u \) is given by the ratio of the mouthpiece pressure to the characteristic impedance of the tube \( Z_0 \). In Norem's formulation the required impedance was the input impedance, but in the present setup the reflections from the bottom of the tube are included in the impulse response of the tube, and the characteristic impedance is the correct choice.

Substituting Eqs. 20 and 24 into Eq. 25, we arrive at

\[ \left( \frac{p_{i}}{Z_0} \right)^3 = B^3(p - p_m)^2 \left[ H + \int_0^t \left[ p(t') - p_{r}(t') \right] h(t - t') dt' \right] \]  \hspace{1cm} (26)

as the nonlinear integral equation which must be solved for \( p_{i} \). This equation can be solved for the value of \( p_{i} \) at any time by knowing the values of \( p_{i} \) at all earlier times. The initial conditions on the solution are set by making all variables zero for all negative times and increasing the blowing pressure \( P \) to its final value at some chosen rate (fast or slow) starting at \( t=0 \). The initial value of \( p_{i} \) calculated from Eq. 26 is used to calculate the values of \( p_{r}, p_{m}, y, \) and finally \( p_{i} \) at a slightly later time using Eqs. 20, 23, 24, and 26. As long as the time interval chosen is sufficiently smaller than the period of the highest frequency component of interest in the solution, the equations should remain well behaved and finally settle to a steady state solution. The transient part of the solution in principle should be correct for the chosen set of parameters of the reed and air column and the specified increase in blowing pressure.
If, in fact this method of solution is plausible, the results should coordinate well with the results discussed in this thesis and in Worman's earlier work.

C. Inclusion of the Bernoulli Force

To include the effects of the Bernoulli force on the reed tip into this formulation, it is only necessary to add the Bernoulli force to the other driving forces on the reed. Thus Eq. 24 should become

\[
y(t) = \int_0^t \left[ P - P_1(t') - p_r(t') - \frac{1}{2} \frac{\rho u_1^2(t')}{y(t') + H} \right] h(t-t') \, dt'
\]  

(27)

In every other way the method of solution is the same.

D. Treatment of the Beating Reed

In order to treat the beating reed realistically it is necessary to know something about the dynamics of the collision between the reed and the facing. If the collision is essentially elastic, the reed will rebound from the facing with the same speed that it had just before the collision. If the collision is completely inelastic, the reed will come to rest against the facing and remain there until the instantaneous pressure difference across the reed is such as to force the reed open again. For partially elastic collisions, the reed will rebound with something less than its incident speed. Bouasse mentions observing both kinds of behavior in different reed instruments. The measurements of McGinnis and Gallagher seem to indicate that the reed does not rebound, but
another investigation is certainly warranted.

The method of calculation for the beating reed is identical to that explained earlier at reed displacements which are small enough that the reed has not yet begun to beat. When the displacement becomes so great that the reed is actually closed (\(y=-H\)), then the method must be changed somewhat. If the collision is considered to be totally inelastic, the past history of the reed motion is forgotten, and the integral in Eq. 26 should be replaced by \(-H\) for all succeeding time intervals until the instantaneous pressure difference across the reed is such as to force it open again. At this point the calculation is continued as before, but with the past values of the factor \([P-p_i(t')-p_r(t')]\) in the integral of Eq. 26 replaced by the minimum constant pressure difference across the reed which would be required to maintain the reed closed against the facing.

If the collision between the reed and mouthpiece is considered to be at least partially elastic, the procedure must be changed somewhat. Simple classical mechanics could be used to determine the initial speed after the rebound, and then to calculate the past history of the displacement appropriate to a linear system corresponding to this velocity and displacement. The driving force as a function of past time necessary to maintain this motion could be inserted in place of the factor \([P-p_i(t')-p_r(t')]\) in the integral of Eq. 26. At this point the calculation continues as before, until the reed again makes contact with the mouthpiece tip. Presumably
this procedure can be used at any oscillation amplitude, and the results should merge smoothly across the point where the reed begins to beat. As the blowing pressure is slowly increased, the reed amplitude increases to the point of beating using the calculation method for the nonbeating reed. When the amplitude grows sufficiently for the reed to begin to beat, the calculation method would change to that outlined above. Obviously the spectrum development will change at the onset of beating. The change should not be abrupt, but continuous as is observed under actual playing conditions.
APPENDIX A

EFFECTS OF THE NONLINEAR REED RESPONSE

It was stated in Sec. III-B that a small nonlinearity in the reed response would make only a second order change in the natural frequency of the reed. To see this, we will assume that the restoring force on the reed is made up of two parts. One of these is the standard harmonic oscillator restoring force which is proportional to the displacement. The other is proportional to the square of the displacement and acts as a hard spring as the reed swings in one direction and as a soft spring as it swings in the opposite direction. Neglecting damping the equation of motion for such an oscillator is

\[
\frac{d^2x}{dt^2} + \omega_0^2 x + \eta x|\dot{x}| = 0 \tag{A1}
\]

where \(\omega_0\) is the natural frequency of the linear oscillator to which this limits at low amplitude, and \(\eta\) is the coefficient of the nonlinearity which is assumed to be small. The exact solution to this equation will be a periodic function of time which can be expanded in a Fourier series. If the nonlinearity is small, then the amplitudes of the higher components will be small, and a decent approximate solution for the present purpose is

\[
x = a + A \cos \omega t + \beta \cos 2\omega t + \gamma \cos 3\omega t \tag{A2}
\]

where \(a, \beta,\) and \(\gamma\) are all much less than \(A\). Substituting this
solution into Eq. A1 and neglecting terms which are third order in the small quantities \( \alpha, \beta, \gamma, \) and \( \eta, \) we find

\[ -\omega^2 [A\cos \omega t + 4\beta \cos 2\omega t + 9\gamma \cos 3\omega t] + \omega_0^2 [\alpha + A \cos \omega t + \beta \cos 2\omega t + \gamma \cos 3\omega t] + \eta [0.5 \omega_0^2 + 2\beta \cos 2\omega t + 2 \alpha A \cos \omega t + \beta A \cos 2\omega t + \gamma A \cos 3\omega t] = 0 \]  

(A3)

Due to the linear independence of sinusoids of different frequencies, this can be divided into four separate equations.

\[ \omega_0^2 \alpha + 2\eta A^2 = 0 \quad (A4a) \]

\[ \omega_0^2 - \omega^2 + 2\alpha \eta + \eta \beta = 0 \quad (A4b) \]

\[ \omega_0^2 \beta - 4\omega_0^2 \beta + 2\eta A^2 + \eta \gamma A = 0 \quad (A4c) \]

\[ \omega_0^2 \gamma - 9\omega_0^2 \gamma + \eta \beta A = 0 \quad (A4d) \]

Solving Eqs. A4a, A4c, and A4d for \( \alpha, \beta, \) and \( \gamma, \) we find

\[ \alpha = -\frac{\eta A^2}{2\omega_0^2} \quad (A5a) \]

\[ \beta = -\frac{\eta A^2}{2(\omega_0^2 - 4\omega^2)} \quad (A5b) \]

\[ \gamma = \frac{\eta^2 A^3}{2(\omega_0^2 - 4\omega^2)(\omega_0^2 - 9\omega^2)} \quad (A5c) \]

Substituting these into Eq. A4b yields

\[ \omega_0^2 - \omega^2 = (\omega_0 - \omega) [\omega_0 + \omega] = \eta^2 A^2 \left( \frac{1}{\omega_0} + \frac{1}{2(\omega_0^2 - 4\omega^2)} \right) \]  

(A6)

Since the frequency shift will be small, \( \omega \) may be replaced by \( \omega_0 \) in both sets of brackets, or

\[ \Delta \omega = \omega_0 - \omega = \frac{7}{12} \frac{\eta^2 A^2}{\omega_0^3} \]  

(A7)

Thus the shift in the natural frequency is second order in the coefficient of nonlinearity \( \eta. \)
APPENDIX B

DETERMINATION OF MINIMUM REED $Q_\tau$ FOR REED REGIME AND OF PLAYING FREQUENCY VERSUS $Q_\tau$

In Sec. III-B it was shown that it is possible for reed instruments to oscillate near the reed frequency if the reed $Q_\tau$ is high enough. This appendix will find the minimum value of $Q_\tau$ for oscillation in the reed regime and will also find the threshold frequency of oscillation as a function of $Q_\tau$ above the minimum $Q_\tau$.

At threshold, the denominators of both expressions in Eq. 3 of Table III must vanish. Thus

$$\frac{\cos \xi_1}{z_1} = A_1 \quad \frac{\sin \xi_1}{z_1} = A_2 \quad (B1)$$

and

$$z_1^{-2} = A_1^{-2} + A_2^{-2} = A^{-2} \quad (B2)$$

Using the expressions for the $A_\perp$ from Table III and the expressions for the reed response parameters $d$ and $\delta$ from Eq. 3 we find

$$A^2 = \left(1-\frac{\omega^2}{\omega^2_r}\right)^2 \frac{\omega^2}{\omega^2_r} + \frac{\omega^2}{\omega^2_Q} + b^2 - 2ab \quad (B3)$$

The magnitude of the total air column input impedance is calculated as the parallel combination of the reed impedance of Eq. 7 and the input impedance of the bore, which beyond cutoff is just the
characteristic impedance $Z_0$ of the tube. This yields

$$|Z|^2 = Z_0^2 \left( \frac{1}{Q_r^2} + \frac{\omega^2}{Q_r} \left( 1 - \frac{\omega^2}{Q_r^2} \right) \right) \left( \frac{w_0^2}{Q_r^2} + \frac{1}{Q_r} \right)^2 + \frac{\omega^2}{Q_r} \left( 1 - \frac{\omega^2}{Q_r^2} \right)$$

(B4)

Combining the previous three equations, eliminating small terms, and solving for $\omega^2/\omega_r^2$ we find

$$\frac{\omega^2}{\omega_r^2} = 1 - \frac{1}{2} \left( Z_0 S_r d_{0 \omega r} - \frac{1}{Q_r} \right)^2 \pm \sqrt{\left( Z_0 a + Z_0 b \right)^2 - Z_0 S_r d_{0 \omega r} + \frac{1}{Q_r^2}}$$

(B5)

In order for a real solution to exist we must have

$$Q_r \geq \left[ Z_0 (a + b - S_r d_{0 \omega r}) \right]^{-1} > 0$$

(B6)

Using Worman's values for all of the required quantities in Eq. B6, we find that oscillation is not possible even with infinite $Q_r$.

This, of course, cannot be realistic in light of the fact that the reed regime does occur on the blowing machine where $Q_r$ is about 12-15. The discrepancy is due to the fact that Worman's value for $S_r$ is unrealistically large. He assumed that the effective reed area is the area of the first 1.1 cm of the reed tip. In fact that area is approximately the entire moving area of the reed. Most of this area moves with a smaller amplitude than the reed tip, and since the effective area is defined as that area which would sweep out the same volume as the reed if it were all moving with the same amplitude as the reed tip, the effective area should be significantly less than the total moving area of the reed. If the reed vibrated as a hinged plate, $S_r$ would be half of the total moving area. Since the reed certainly has some curvature as it swings back and forth, $S_r$ is less
than half the total moving area. Using a value of $S_r$ which is slightly less than half of Worman's value, we find that oscillation in the reed regime is possible with minimum $Q_r$ in the range of 10 to 15. Table V shows the minimum $Q_r$ for several values of $S_r$.

At the threshold $Q_r$ we find that

$$\frac{\omega}{\omega_r} = [1-\frac{1}{3}(Z_0a+Z_0b)^2]^\frac{3}{5} = 1-\frac{1}{3}(Z_0a+Z_0b)^2 = .98 \quad (57)$$

At higher values of $Q_r$ the frequency is further lowered from its value with the minimum $Q_r$, but even if $Q_r$ were 50 per cent above its minimum value, the reed regime would be only about 10 per cent less than the reed natural frequency.
APPENDIX C

DETERMINATION OF CHANGE IN PLAYING FREQUENCY
WITH CHANGE IN REED NATURAL FREQUENCY

In determining the change in playing frequency which occurs with a given change in reed natural frequency, we will make a number of simplifying assumptions. First we will neglect the fact that the requirements for energy input to the oscillating system require that the playing frequency be slightly below the frequency of the air column impedance maximum. The frequency shift to be calculated is the shift in the impedance maximum itself. In addition the damping of both the reed and the air column are neglected in this discussion. For the air column this is reasonable because its $Q$ is about 30, and the shift in the natural frequency caused by including the damping is quite small. While the $Q_r$ of the reed is much smaller, the frequencies considered are far below its resonance, and the inclusion of the effects of damping would again make very little difference in the solution. It must be remembered that we are in fact calculating the derivative of the playing frequency with respect to reed frequency, and just because the effects of these parameters on the playing frequency are small does not necessarily mean that their effects on the derivative are also small. However computer calculations which include the effects of damping agree with this approximation within a few per cent. This accuracy is acceptable for the present purpose.
If the input impedance of the bore is taken to be that of a cylindrical tube with characteristic impedance $Z_0$, we have

$$Z_b = -iz_0 \tan \frac{\omega_1}{c}$$  \hspace{1cm} (C1)

Calculating the total impedance as the parallel impedance of $Z_b$ and $Z_r$ given in Eq. 7 but with $s_r$ set equal to zero yields

$$|Z|^2 = \frac{Z_0 \mu_r (\omega_r^2 - \omega^2)^2 \tan k l}{[\mu_r (\omega_r^2 - \omega^2)^2 - s_0 \omega \tan k l]^2}$$  \hspace{1cm} (C2)

This impedance will be maximum when the denominator is zero, or when

$$\tan k l = \frac{\mu_r}{\omega Z_0 s_r} (\omega_r^2 - \omega^2)^2$$  \hspace{1cm} (C3)

Unless the playing frequency is very near to the reed frequency, the right side of this equation is much greater than unity, and the playing frequency will be near that for which $\tan k l$ is infinite. If we define $\omega_0$ by

$$\frac{\omega_0^1}{c} = \frac{\pi}{2}$$  \hspace{1cm} (C4)

then we may replace the tangent function in Eq. C3 by \(\frac{c}{(\omega_0 - \omega)l}\), and the equation to be solved for the playing frequency can be written as

$$\Delta \omega = \frac{\omega_0 - \omega}{\omega} = \frac{Z_0 S c}{\mu_r l (\omega_r^2 - \omega^2)}$$  \hspace{1cm} (C5a)

or

$$\frac{\omega^2 - \omega_r^2}{\omega_r^2} = \frac{Z_0 S c}{\mu_r l \omega_0 - \omega}$$  \hspace{1cm} (C5b)

Taking the differential of both sides of Eq. C5b yields
\[
2\omega \frac{d\omega}{r} - 2\omega d\omega = \frac{Z_0 S c}{\mu_r \frac{1}{r}} \left( \frac{d\omega}{\omega_0 - \omega} + \frac{\omega d\omega}{(\omega - \omega_0)^2} \right)
\]  

(C5)

Solving this for the fractional change in reed frequency over the fractional change in playing frequency

\[
\frac{d\omega/\omega}{d\omega_r/\omega_r} = \frac{Z_0 S c}{\mu_r \frac{1}{r}(\omega_r^2 - \omega^2)} = \frac{2k\omega}{\omega}
\]

(C7)

where Eq. C5a has been used to simplify the expression. This states that the fractional change in playing frequency divided by the fractional change in reed frequency is approximately equal to twice the percentage change in playing frequency caused by the elastic reed termination. Putting in the values for the parameters in Eq. C7 gives a fractional frequency change of about 4 per cent when the reed frequency matches the second harmonic of the playing frequency and about 2 per cent when the reed frequency is near the third harmonic. This means that in order to change the playing frequency by 10 cents (0.6 per cent), the reed frequency must be changed by about 15 per cent if it is near the second harmonic of the playing frequency and by 30 per cent if it is near the third harmonic.
# TABLE I

OFTEN USED SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>...</td>
<td>Pressure difference across reed</td>
</tr>
<tr>
<td>P</td>
<td>...</td>
<td>Blowing pressure in mouth</td>
</tr>
<tr>
<td>$p_m$</td>
<td>$P-p$</td>
<td>Mouthpiece pressure</td>
</tr>
<tr>
<td>$p_i, p_m$</td>
<td>...</td>
<td>See p. 69</td>
</tr>
<tr>
<td>Z</td>
<td>...</td>
<td>Total air column input impedance</td>
</tr>
<tr>
<td>$Z_b$</td>
<td>...</td>
<td>Input impedance of bore</td>
</tr>
<tr>
<td>$Z_r$</td>
<td>...</td>
<td>Reed impedance</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>$2.31 \times 10^6$ kg/m$^4$ sec</td>
<td>Characteristic impedance of tube</td>
</tr>
<tr>
<td>$d_0$</td>
<td>$8.01 \times 10^{-3}$ m$^3$/nt</td>
<td>Reed compliance</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>$2.31 \times 10^{-2}$ kg/m$^2$</td>
<td>Reed effective mass density</td>
</tr>
<tr>
<td>$g_r$</td>
<td>$2.91 \times 10^3$ rad/sec</td>
<td>Reed damping coefficient</td>
</tr>
<tr>
<td>$* S_r$</td>
<td>$7.30 \times 10^{-5}$ m$^2$</td>
<td>Reed effective area</td>
</tr>
<tr>
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<td>...</td>
<td>Reed quality factor</td>
</tr>
<tr>
<td>$* w_{r,f}$</td>
<td>2-3 kHz</td>
<td>Reed natural frequency</td>
</tr>
<tr>
<td>H</td>
<td>$4 \times 10^{-3}$ m</td>
<td>Equilibrium reed opening</td>
</tr>
<tr>
<td>y</td>
<td>...</td>
<td>Reed displacement from equilibrium</td>
</tr>
<tr>
<td>$c,v$</td>
<td>345 m/sec</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>B</td>
<td>0.08 (SI units)</td>
<td>Flow coefficient (See p. 35)</td>
</tr>
<tr>
<td>$F'_{ij}$</td>
<td>...</td>
<td>See Table II</td>
</tr>
</tbody>
</table>

* These values have been changed from those of Wormald
TABLE II

VALUES OF THE $F_{ij}'$

\[ F_{00}' = 3.00 \times 10^{-5} \text{ } m^3/\text{sec} \]
\[ F_{01}' = 2.15 \times 10^{-7} \text{ } m \text{ sec/kg} \]
\[ F_{10}' = 3.18 \times 10^{-1} \text{ } m^2/\text{sec} \]
\[ F_{02}' = -2.46 \times 10^{-11} \text{ } m^5 \text{ sec}^3/\text{kg}^2 \]
\[ F_{11}' = 5.13 \times 10^{-4} \text{ } m^3 \text{ sec/kg} \]
\[ F_{20}' = 2.01 \times 10^{-2} \text{ } m/\text{sec} \]
\[ F_{03}' = 1.23 \times 10^{-15} \text{ } m^6 \text{ sec}^5/\text{kg}^3 \]
\[ F_{12}' = -2.56 \times 10^{-8} \text{ } m^4 \text{ sec}^3/\text{kg}^2 \]
\[ F_{21}' = 1.60 \times 10^{-1} \text{ } m^2 \text{ sec/kg} \]
\[ F_{30}' = -3.34 \times 10^{5} \text{ sec}^{-1} \]

These values were calculated and initially reported by Worman. See Ref. 2, pp. 56-58.
TABLE III

NONLINEAR EQUATIONS OF THE OSCILLATING SYSTEM

\[ p - p_0 = G_{01} + G_{02} p_0 + G_{03} p_0^2 + G_{04} p_0^2 + G_{05} p_0^2 + G_{06} p_0^2 + G_{07} p_0^3 + G_{08} p_0^2 \cdot \cdots \]

\[ p_1 = \frac{a_0 p_0 p_2 + a_1 p_0 p_3 + \cdots}{\frac{\cos \gamma_1}{z_1} - A_1 (d_1, \delta_1)} = \frac{b_0 p_0 p_2 + b_1 p_0 p_3 + \cdots}{\frac{\sin \gamma_1}{z_1} - A_2 (d_1, \delta_1)} \]

\[ p_n \cos \phi_n = \frac{p_1 b_n (d_1, \delta_1)}{\left\{ \left( \frac{\cos \gamma_n}{z_n} - A_1 (d_n, \delta_n) \right)^2 + \left( \frac{\sin \gamma_n}{z_n} - A_2 (d_n, \delta_n) \right)^2 \right\}^{\frac{1}{2}}} \]

\[ p_n \sin \phi_n = \frac{p_1 c_n (d_1, \delta_1)}{\left\{ \left( \frac{\cos \gamma_n}{z_n} - A_1 (d_n, \delta_n) \right)^2 + \left( \frac{\sin \gamma_n}{z_n} - A_2 (d_n, \delta_n) \right)^2 \right\}^{\frac{1}{2}}} \]

\[ A_1 (d, \delta) = \left( F_{10} - 2 F_{10} p_0 d_0 + F_{11} p_0 \right) d \cos \delta - \left( F_{01} - F_{11} p_0 d_0 + 2 F_{02} p_0 \right) \]

\[ A_2 (d, \delta) = \left( F_{10} - 2 F_{10} p_0 d_0 + F_{11} p_0 \right) d \cos \delta \]

The \( G_{0i} \) are the same as Worman's \( C_{0i} \).
### TABLE IV

**RESULTS OF BLOWING MACHINE EXPERIMENT**

<table>
<thead>
<tr>
<th>Clarion Register Note</th>
<th>Reed Regime Note</th>
<th>Component Number Matched</th>
<th>Cents Deviation</th>
<th>Per Cent Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₄₋38¢</td>
<td>A₇₋20¢</td>
<td>6</td>
<td>+18</td>
<td>+1.07</td>
</tr>
<tr>
<td>B♭₆+20¢</td>
<td>F₇+65¢</td>
<td>3</td>
<td>+45</td>
<td>+2.68</td>
</tr>
<tr>
<td>C₆+30¢</td>
<td>C₇+32¢</td>
<td>2</td>
<td>+2</td>
<td>+0.12</td>
</tr>
<tr>
<td>C₆+8¢</td>
<td>G₇+4¢</td>
<td>3</td>
<td>−4</td>
<td>−0.24</td>
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<tr>
<td>C♯₆+20¢</td>
<td>G♯₇+40¢</td>
<td>3</td>
<td>+20</td>
<td>+1.19</td>
</tr>
<tr>
<td>D♯₆₋25¢</td>
<td>D♯₇₋10¢</td>
<td>2</td>
<td>+15</td>
<td>+0.89</td>
</tr>
<tr>
<td>E₆+10¢</td>
<td>E₇+0¢</td>
<td>2</td>
<td>−10</td>
<td>−0.59</td>
</tr>
</tbody>
</table>

### TABLE V

**MINIMUM Qᵣ FOR REED REGIME FOR SEVERAL VALUES OF Sᵣ**

<table>
<thead>
<tr>
<th>Sᵣ</th>
<th>Qᵣ,min</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2x10⁻⁵</td>
<td>∞</td>
</tr>
<tr>
<td>7.3x10⁻⁵</td>
<td>16</td>
</tr>
<tr>
<td>7.0x10⁻⁵</td>
<td>13</td>
</tr>
<tr>
<td>6.5x10⁻⁵</td>
<td>10</td>
</tr>
<tr>
<td>6.0x10⁻⁵</td>
<td>8.4</td>
</tr>
</tbody>
</table>
Figure 2

Constant Pressure Air Source
Player's Lungs and Mouth

Reed Flow Control Valve
\[ u^{1/2} B p^{2/3} (y+H)^{4/3} \]

Reed Acts as Resonant Air Column Termination

Flow Divides Into Two Parts
1. Goes into Bore
2. Fills Space Vacated by Moving Reed
FIGURE 4

MEASURED INPUT IMPEDANCE

B♭ CLARINET PLAYING WRITTEN C₄

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)

1000 2000 3000

x10
FIGURE 6

IDEALIZED CLARINET INPUT IMPEDANCE

INPUT IMPEDANCE (LINEAR SCALE)

NORMALIZED FREQUENCY ($f/f_0$)

REEF FREQUENCY
FIGURE 7

Z AND $Z_{\text{min}}$ VERSUS FREQUENCY
REED REGIME POSSIBLE

INPUT IMPEDANCE
(LINEAR SCALE)

$Z_{\text{min}} = \frac{1}{A}$

NORMALIZED FREQUENCY ($f/f_r$)

0.25  0.50  0.75  1.00  1.25
FIGURE 8

Z AND Z$_{\text{min}}$ VERSUS FREQUENCY

REED REGIME NOT POSSIBLE

INPUT IMPEDANCE (LINEAR SCALE)

$Z_{\text{min}} = \frac{1}{A}$

NORMALIZED FREQUENCY ($f/f_r$)
FIGURE 9

REED TRANSCONDUCTANCE

MAGNITUDE

$\frac{n}{f_r}$

0.25 0.50 0.75 1.00 1.25

$A, A_1, A_2$

PHASE

$\frac{\pi}{2}$

$\alpha = \tan^{-1}(A_2/A_1)$

$\frac{n}{f_r}$

0 0.25 0.50 1.00 1.25

$-\frac{\pi}{2}$
Figure 10

Input Impedance

\[ Z = z e^{-i\xi} \]

Magnitude (Linear Scale)

Normalized Frequency \( (f/f_r) \)

Phase

\( \frac{\pi}{2} \)

\( 0 \)

\( 0.25 \)

\( 0.75 \)

\( 1.00 \)

\( 1.25 \)
FIGURE 12

TONIC HOLE LATTICE
(STI)

2a = 15 mm
2b = 5.2 mm
2S = 33.7 mm
\( t = 7.6 \text{ mm} \)

21 HOLES PER ROW
3 ROWS SPACED 120° AROUND TUBE AXIS

APPROX. HALF SCALE
FIGURE 14

INPUT IMPEDANCE OF CYLINDRICAL TUBE WITH THL 1500 TERMINATION
- FIBERGLAS DAMPING REMOVED -

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)
FIGURE 15

INPUT IMPEDANCE OF CYLINDRICAL TUBE WITH THL1000 TERMINATION

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)

x10
FIGURE 16

INPUT IMPEDANCE OF CYLINDRICAL TUBE WITH THL1500 TERMINATION

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)

1000 2000 3000

x10
FIGURE 17

INPUT IMPEDANCE OF CYLINDRICAL TUBE WITH THL 2000 TERMINATION

FREQUENCY (Hz)

1000

2000

3000

(Linear Scale)

Input Impedance
FIGURE 18

TOP JOINT
(ST2)

$2a = 15 \text{ mm}$
$2b = 5.2 \text{ mm}$
$2S = 17.2 \text{ mm}$
$t = 7.6 \text{ mm}$

APPROX. HALF SCALE
FIGURE 19

INPUT IMPEDANCE OF TOP JOINT
WITH THL1500 TERMINATION
-ALL HOLES OPEN-

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)

x10
FIGURE 20

INPUT IMPEDANCE OF TOP JOINT WITH THL 1500 TERMINATION —FOUR HOLES CLOSED—

INPUT IMPEDANCE (LINEAR SCALE)

FREQUENCY (Hz)

1000 2000 3000
FIGURE 21

INPUT IMPEDANCE OF TOP JOINT WITH THL1500 TERMINATION - ALL HOLES CLOSED

FREQUENCY (Hz)

INPUT IMPEDANCE

LINEAR SCALE
FIGURE 22

STRAIN GAUGE

STRAIN GAUGE DIVIDER

ALL PASS FILTER

POWER AMP

BARREL

TOP JOINT (ST2)

THL1500 (ST1)

HORN DRIVER

REED RANGE EXPERIMENT
FIGURE 23
INTERNAL SPECTRA OF G₅

RELATIVE AMPLITUDE (DB)

1 2 3 4 5 6
COMPONENT NUMBER

-40 -30 -20 -10 0

● REGIME OPTIMIZED
○ 10 CENTS FLATTER
X 10 CENTS SHARPER
FIGURE 24
INTERNAL SPECTRA OF A₆

RELATIVE AMPLITUDE (DB)

0
-10
-20
-30
-40

1 2 3 4 5 6 COMPONENT NUMBER

○ REGIME OPTIMIZED
○ 10 CENTS FLATTER
X 10 CENTS SHARPER
FIGURE 25

INTERNAL SPECTRA OF $D_6^#$

- Relative Amplitude (dB)
- Component Number

- **Regime Optimized**
- ○ 10 Cents Flatter
- X 10 Cents Sharper
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    See especially Fig. 5.


43. Worman, pp. 19,57.

44. Worman, pp. 121-123.


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