Analog model for thermoviscous propagation in a cylindrical tube

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Modeling acoustic propagation in tubes including the effects of thermoviscous losses at the tube walls is important in applications such as thermoacoustics, hearing aids, and wind musical instruments. Frequency dependent impedances for a tube transmission line model in terms of the so-called thermal and viscous functions are well established, and form the basis for frequency domain analysis of systems that include tubes. However, frequency domain models cannot be used for systems in which significant nonlinearities are important, as is the case with the pressure-flow relationship through the reed in a woodwind instrument. This paper describes a cylindrical tube model based on a continued fraction expansion of the thermal and viscous functions. The model can be represented as an analog circuit model which allows its use in time domain system modeling. This model avoids problems with fractional derivatives in the time domain.

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I. INTRODUCTION

This paper derives an analog model for acoustic propagation in a rigid tube with a circular cross section as shown in Fig. 1, including the effects of thermal and viscous losses at the walls of the tube. There is a long history of the analysis of propagation in tubes including thermal and viscous effects, starting with Zwikker and Kosten,1 and well summarized by Stinson.2 The present analysis includes a continued fraction expansion of the thermal and viscous functions that allows implementation as a set of analog ladder networks. The approximation is accurate within a known finite frequency band determined by the number of terms in the continued fraction. Implementation as an analog network avoids problems with fractional derivatives in the time domain.3 Note that a different use of continued fraction expansions was made by Kergomard4,5 that is more efficient than the one derived here when it can be used. However, the Kergomard model does not allow arbitrary impedance termination at the far end of the tube.

Acoustic propagation modeling in cylindrical tubes continues to be an active area of investigation, for example for the study of absorbing materials,6 for propagation in small tubes such as in hearing aids,7,8 for propagation at very low frequencies,9,10 for thermoacoustics,11 and for modeling of wind musical instruments.5 In many of these applications, the systems are sufficiently linear that a frequency domain analysis provides the required accuracy. However, in cases where nonlinear effects are significant, a time domain model is needed and the tube model described in this paper may prove useful.

One such case is a simplified model of oscillation in a reed woodwind musical instrument such as the clarinet. The air column can be modeled as a linear transmission line of cascaded thermoviscous tube segments interspersed with short branch segments that are the closed or open tone holes. The reed can be modeled as a linear harmonic oscillator whose tip position defines the dimension of the opening of a small orifice that allows air to flow into the air column. The volume flow of air through the reed opening is a nonlinear function of the pressure difference across the reed. The two linear systems (the air column and the reed) are coupled by the nonlinear flow through the reed as follows: The oscillating pressure in the mouthpiece drives the motion of the reed. The motion of the reed modulates the air flow into the tube at the playing frequency in a way that adds energy to the standing wave in the air column. A more complete description of this system is given in Nederveen12 and in Fletcher and Rossing.13 A time domain system model of this type using the methods in this paper is in development by one of the authors.

II. TUBE MODEL WITH THERMOVISCOUS LOSSES

The acoustic variables in this analysis are pressure $p$ and volume velocity $u$. The subscripts shown in Fig. 1 indicate the values of the variables at the two ends of the tube. Using the language of Benade,14 Keefe,16 and Swift,11 the circular acoustic transmission line has the series impedance $Z$ per unit length and shunt admittance $Y$ per unit length given by

$$Z = \frac{j\omega p}{\pi a^2 (1 - f_o)}.$$  

\[ (1) \]
where the medium has density \( \rho \) and sound speed \( c \), \( \gamma \) is the ratio of specific heats, \( \omega \) is the angular frequency of the wave, and the tube has radius \( a \). (Note that the Benade\textsuperscript{14} reference contains errors that were first identified by Backus\textsuperscript{15} and later corrected by Keefe.\textsuperscript{16} Those errors do not affect the parts of the reference that are used in the present case.) For any tube geometry, the thermal and viscous functions \( f_t \) and \( f_v \) are of the same form. For cylindrical tubes they are given by

\[
 f_{v,t} = \frac{2f_1(z_{v,t})}{z_{v,t}f_0(z_{v,t})} \quad \text{with} \quad z_{v,t} = (j-1)\frac{a}{\delta_{v,t}} ,
\]

where \( \delta_v \) and \( \delta_t \) are the viscous and thermal boundary layer thicknesses, respectively, given by

\[
 \delta_v = \sqrt{\frac{2\nu}{\omega}}, \quad \delta_t = \sqrt{\frac{2\kappa}{\omega \rho C_p}},
\]

and \( \nu \) is the kinematic viscosity of the fluid, \( \kappa \) is its thermal conductivity, and \( C_p \) is its specific heat at constant pressure. With these definitions, the infinitesimal tube of length \( d\ell \) can be represented by the impedance network shown in Fig. 2. Instead of this \( T \) network configuration, it would be possible to reduce the component count in the computations by using a \( \Gamma \) or reversed-\( \Gamma \) configuration in which one of the series impedances is eliminated and the value of the other is doubled. However, the \( T \) network is a centered finite difference approximation that is accurate to second order in \( d\ell \) while either \( \Gamma \) network is a one-sided finite difference approximation accurate only to first order. Furthermore, in most cases several sections of either network would be cascaded, as shown below in Fig. 10. This reduces the effect of both the errors and the difference in component count.

A tube of finite length would require a number of cascaded sections of Fig. 2. For lossless transmission, this cascaded network is not the most efficient method for analysis. The method first described by Brinian for electrical transmission lines\textsuperscript{17} is much faster and has been implemented in many computational codes. Furthermore, when a transmission line includes frequency independent losses, a number of similar methods are appropriate.\textsuperscript{18–20} All of these methods can be used in either time domain or frequency domain analyses. However, when the losses are frequency dependent, as are the thermoviscous losses in Eqs. (1) and (2), a different method is necessary. Special purpose transmission line modeling methods,\textsuperscript{21} including methods that model thermoviscous losses\textsuperscript{22} are possible. However, the present method has the advantage that it can be implemented in a wide range of generally available computational codes.

The determination of the segment length \( d\ell \) that provides sufficient accuracy for a particular modeling application has been discussed, for example, by Christopoulos.\textsuperscript{23} This reference suggests that ten segments per wavelength for the highest frequency of interest may often be sufficient, and provides guidance for calculating the error. For the present case, Sec. II C describes a calculation in the frequency domain that shows the effects of the finite segmentation that allows a quantitative evaluation of the error due to segmentation. Note also that, in addition to the dispersion caused by frequency dependent propagation,\textsuperscript{24} an additional dispersion\textsuperscript{25} is caused by the finite segment length \( d\ell \). Correct estimation of dispersion may be especially important in applications that require the accurate calculation of the acoustic resonance frequencies of a system.

This section will develop a complete model for the tube with thermoviscous losses. The development starts by finding models for the impedance per unit length \( Z \) and the admittance per unit length \( Y \), and then assembling the finite length of tube as the cascade of a number of short segments.

### A. The impedance per unit length \( Z \)

Equation (1) can be rewritten as

\[
 Z d\ell = \frac{j\omega pd\ell}{\pi a^2 (1 - f_v)} = \frac{j\omega pd\ell}{\pi a^2} \left( \frac{1 - f_v}{1 - f_v} \right)
\]

\[
 = \frac{j\omega pd\ell}{\pi a^2} \left( 1 - \frac{1}{1 - f_v} \right)^{-1}
\]

\[
 = \frac{j\omega pd\ell}{\pi a^2} \left( 1 - \frac{2f_1(z_{v,t})}{z_{v,t}f_0(z_{v,t})} \right)^{-1} .
\]

Note in Eq. (6) that the trivial manipulation of the term in parentheses significantly simplifies the subsequent derivation. The Bessel function ratio in Eq. (7) can then be expanded as a continued fraction\textsuperscript{26} and simplified to

FIG. 2. The impedance network representation of a tube of infinitesimal length.


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\[ Zd\ell = j\omega L_0 + R_v \]
\[ + \frac{1}{j\omega L_0 + \frac{1}{2R_v}} \]
\[ + \frac{1}{j\omega L_0 + \frac{1}{3R_v}} \]
\[ + \frac{1}{j\omega L_0 + \frac{1}{4R_v}} \]
\[ + \frac{1}{j\omega L_0 + \frac{1}{5R_v}} \]
\[ + \frac{1}{j\omega L_0 + \frac{1}{6R_v}} + \cdots \]
\[ (8) \]

where \( L_0 = \rho d/\pi a^2 \) and \( R_v = 8\pi \rho /a d/(\pi a^2)^2 \). Equation (8) is a computationally efficient method of calculating the thermoviscous impedance in the frequency domain. However it also lends itself to an implementation in the time domain. This impedance can be implemented as an analog circuit by the infinite ladder network shown in Fig. 3. As an analog circuit, it is straightforward to implement this impedance in a time domain model using techniques that were first developed as the SPICE model for electrical circuits,\textsuperscript{27,28} and have subsequently been implemented in other computational frameworks.\textsuperscript{29,30}

Figure 4 shows the real part of the impedance per unit length for a tube with a diameter of 1.5 cm. The “analytical” curve is calculated from Eq. (7) using MATLAB with its full numerical precision, and the approximations are calculated with the circuit of Fig. 3 using the LTspice\textsuperscript{31} implementation of the SPICE code. The curves for each analog circuit are labeled with the number of circuit “branches” or rungs of the ladder network, where each branch adds one resistance and one ineritance to the network. Figure 3 shows a circuit with six branches. Figure 4 shows that the analog circuit approximation is accurate at low frequencies and diverges from the correct solution at high frequencies. The approximation is extended to a higher frequency range as more branches are added to the circuit. Using a circuit with 24 branches (49 components) gives a reasonably good approximation to more than 4 orders of magnitude in frequency above the transition frequency that is approximately 1 Hz for this tube.

Figure 5 shows the fractional error in the continued fraction approximation for the real part of the series impedance compared to the calculation with full numerical precision. The fractional error is the magnitude of the difference between the exact value [Eq. (7)] and the analog circuit approximation [Eq. (3)] divided by the exact value. With 24 branches, the approximation for this tube is accurate within 2% for frequencies below 10 kHz and to 0.1% below 2 kHz.

B. The admittance per unit length \( Y \)

Consider next the admittance per unit length in Eq. (2). In a way that is similar to the derivation of Eq. (7), the thermal function \( f_t \) can be expanded as a continued fraction,\textsuperscript{26} as

\[ Yd\ell = \frac{j\omega a^2 d\ell}{\rho c^2} \left[ 1 + (\gamma - 1)f_t \right] \]
\[ = \frac{j\omega a^2 d\ell}{\rho c^2} \left[ \frac{j\omega a^2 d\ell(\gamma - 1)}{\rho c^2} \frac{2J_1(z)}{J_0(z)} \right] \]
\[ (9) \]
where $C_0 = \pi \alpha^2 \ell \mu c^2$, $C_1 = (\gamma - 1) C_0$, and $R_t = \rho^2 c^2 C_p / 8\pi k (\gamma - 1) \ell$. This admittance can be implemented as the infinite ladder network shown in Fig. 6.

Figure 7 shows the real part of the admittance per unit length for a tube with a diameter of 1.5 cm. The analytical curve is calculated with Eq. (9) using MATLAB with its full numerical precision, and the approximations are calculated with the analog circuit of Fig. 6 using LTspice.\(^{31}\) As above, the curves for the analog circuit are labeled with the number of circuit branches of the ladder network. Again, the approximation is extended to a higher frequency range as more branches are added to the circuit. Using a circuit with 24 branches (49 components) gives a reasonably good approximation to more than 4 orders of magnitude in frequency above the adiabatic transition frequency that is somewhat below 1 Hz for this tube.

Figure 8 shows the fractional error in the continued fraction approximation compared to the calculation with full numerical precision. The fractional error is the magnitude of the difference between the exact value [Eq. (10)] and the analog circuit approximation [Eq. (9)] divided by the exact value. With 24 branches, the approximation is accurate within 2\% for frequencies below 10 kHz and to 0.1\% below 2.5 kHz.

### C. The finite thermoviscous tube model

A complete model of the infinitesimal tube section combines the models for $Z$ and $Y$ into the impedance network of Fig. 2, as shown in Fig. 9. The number of branches in each leg, shown here as three, should be extended to achieve the required bandwidth, as described in Secs. II A and II B. Note that the component values in the series impedance legs are changed by a factor of 2 from Fig. 3 as indicated in Fig. 2.

A tube of finite length $\ell$ can then be modeled by a number of cascaded segments of Fig. 9, or by the slightly simpler configuration of Fig. 10. In a practical implementation of this model, of course, only a finite number $N$ of cascaded sections can be used. The effects on the accuracy calculation of these finite approximations are shown in Fig. 11 for a tube that has a length of 25 cm and a diameter of 1.5 cm. Figure 11 shows the magnitude of the calculated input impedance at one end of the tube with the other end terminated with zero impedance. The bandwidth of accurate approximation is increased by using a greater number of shorter sections.
this tube, the values of the resonance and antiresonance frequencies are within 2% when the length of the cascaded sections is 10% of a wavelength. This resolution must be considered in every particular application. For example, the suggested 2% accuracy in resonance frequency may be sufficient in many applications, but it would not be sufficient to calculate the playing frequency of a musical instrument.

III. CONCLUSION

The approximation presented in this paper provides arbitrary accuracy over any defined finite bandwidth, within the limits of the numerical accuracy of the machine that is employed for the computation. The continued fraction computation is generally robust in a numerical computation as it does not result in the subtraction of approximately equal numbers. The apparent disadvantage when this is implemented in an analog circuit computation is the large number of components required to model a single tube. For example, the tube in the third graph in Fig. 11 includes approximately 4750 components and a matrix size to solve at each frequency, or each time step, of 2424. While not insignificant, this is not an especially large problem size for modern computer hardware and software.

FIG. 11. (Color online) Magnitude of the acoustic input impedance of a tube 25 cm long and 1.5 cm in diameter, open at the opposite end. Dashed curves are calculated using the exact analytical expression. The solid curves are the analog circuit approximation, with 10, 20, and 40 branches in the circuit.


Reference 16, Fig. 4.

Reference 23, Sec. 4.4.


Information on Spice can be found at http://bwrc.eecs.berkeley.edu/Classes/ICBook/SPICE/ (Last viewed November 21, 2013).

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The LTspice program is freely available from Linear Technology Corp. through their web site at http://www.linear.com (Last viewed November 21, 2013).